Compact Sets

MATH 409 HNR Analysis on the Real Line, Texas A&M University

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Recall the definition of compact sets discussed:

Definition 0.1: Compact Sets

A set $S \subseteq \mathbb{R}$ is **compact** if every sequence in S has a convergent subsequence with limit in S.

This definition is enough for $\mathbb{R}$, but there is a generalized definition.

1 Metric Spaces

Compact sets are generally defined on *metric spaces*. What is this?

Definition 1.1: Metric Function

Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric function** on X if

- $d(x, y) > 0$ for $\forall x,\ y \in X$ with $x \neq y$
- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$ for $\forall x,\ y \in X$
- $d(x, y) \leq d(x, z) + d(z, y)$ for $\forall x,\ y,\ z \in X$.

Definition 1.2: Metric Space

A nonempty set X with a metric function $d : X \times X \to \mathbb{R}$ is called a metric space.

Example 1

$\mathbb{R}$ is a metric space with a metric function $d(x, y) = |x - y|$. We claim that the metric $d(x, y) = |x - y|$ satisfies the four axioms of a metric function.

- If x, $y \in \mathbb{R}$ with $x \neq y$, then $|x - y| > 0$.
- $|x - x| = 0$, and if $|x - y| = 0$ then $x = y$
- For all x, $y \in \mathbb{R}$, $|x - y| = |y - x|$.
- For all x, y, $z \in \mathbb{R}$, $|x - y| \leq |x - z| + |z - y|$ (triangle inequality).

Therefore, $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space.

Example 2

Let X be the set of all bounded real-valued functions on $A(\neq \emptyset)$. For f, $g \in X$, we define $d(f, g) = \sup \{|f(x) - g(x)| : x \in A\}$. Since

- $0 \leq |f(x) - g(x)| \leq |f(x)| + |g(x)| \leq 2M$ for all $x \in A$
- $d(f, g) = 0 \Leftrightarrow f = g$ since $|f(x) - g(x)| \leq d(f, g)$ for $\forall x \in A$
- $d(f, g) = d(g, f)$
- $\sup\{|f(x) - g(x)| : x \in A\} = \sup\{|f(x) - h(x) + h(x) - g(x)| : x \in A\} \leq \sup\{|f(x) - h(x)| : x \in A\} + \sup\{|h(x) - g(x)| : x \in A\} = d(f, h) + d(h, g)$,

d is a metric on X.

With this definition, we can generalize what we have done on $\mathbb{R}$ to a metric space.

Definition 1.3: Neighborhood

Let (X, d) be a metric space. For $\epsilon > 0$ and $p \in X$, the set

$$N_\epsilon(p) = \{x \in X : d(p, x) < \epsilon\}$$

is called an ϵ-**neighborhood** of p.

Definition 1.4: Bounded

Let (X, d) be a metric space. A set $E \subseteq X$ is bounded if there is $x_0 \in X$ and $M \in \mathbb{R}$ such that for all $x \in E$, $d(x, x_0) < M$.

2 Open and Closed Sets

From now on, denote X as a metric space. The proofs for results are omitted if they have the exact same proof with the case $X = \mathbb{R}$.

Definition 2.1: Interior Point

Let $E \subseteq X$ be a set. A point p is an **interior point** of E if for some $\epsilon > 0$, the ϵ-neighborhood of p is entirely in E. That is, $N_\epsilon(p) \subset E$.

Definition 2.2: Isolated Point

Let $E \subseteq X$ be a set. A point p is an **isolated point** of E if for some $\epsilon > 0$, p is the only point of E in the ϵ-neighborhood of p. That is, $N_\epsilon(p) \cap E = \{p\}$.

Definition 2.3: Boundary Point

Let $E \subseteq X$ be a set. A point p is a **boundary point** of E if any neighborhood of c contains points from both E and $E^{\complement}$. That is, $N_\epsilon(p) \cap E \neq \emptyset$ and $N_\epsilon(p) \cap E^{\complement} \neq \emptyset$.

Definition 2.4: Limit Point

Let $E \subseteq X$ be a set. A point p is a **limit point** of E if any ϵ-neighborhood of p contains a point of E other than p. That is, $N_\epsilon(p) \cap (E \setminus \{p\}) \neq \emptyset$.

Proposition.

p is a limit point of E if and only if there is a sequence of elements in $E \setminus \{p\}$ converging to p.

Proof. ($\Leftarrow$) Suppose the sequence $\{x_n\}$ of elements of $E \setminus \{p\}$ converges to p. Then $\forall \epsilon > 0$, $\exists N$ such that $\forall n \geq N$, $d(x_n, p) < \epsilon$. Thus p is a limit point of E.

($\Rightarrow$) If p is a limit point of E, choose $x_n \in S \setminus \{c\}$ with $d(x_n, p) < 1/n$, then $x_n \to p$. ∎

Definition 2.5: Open and Closed Sets

A set $S \subseteq \mathbb{R}$ is **open** if every point of S is an interior point. A set $S \subset \mathbb{R}$ is **closed** if it contains all of its limit points.

Note that the definitions coincide to the definitions discussed in class if we set $X = \mathbb{R}$.

Theorem 2.1

- A complement of an open set is closed.
- A complement of a closed set if open.

Theorem 2.2

Let (X, d) be a metric space.

1. If $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is a collection of open sets of X, then $\bigcup_{\alpha \in A} \mathcal{O}_\alpha$ is open. That is, an arbitrary union of open sets is open.
2. If $\{\mathcal{O}_1, \cdots, \mathcal{O}_n\}$ is a finite collection of open sets of X, then $\bigcap_{j=1}^{n} \mathcal{O}_j$ is open. That is, a finite intersection of open sets is open.

Proof. (1) We may assume that $\bigcup \mathcal{O}_\alpha \neq \emptyset$. Let $p \in \bigcup_{\alpha \in A} \mathcal{O}_\alpha$, then $p \in \mathcal{O}_\alpha$ for some $\alpha \in A$. Since $\mathcal{O}_\alpha$ is open, $\exists \epsilon > 0$ such that $N_\epsilon(p) \subseteq \mathcal{O}_\alpha \subseteq \bigcup \mathcal{O}_\alpha$. Thus, p is an interior point of $\bigcup_{\alpha \in A} \mathcal{O}_\alpha$.

(2) Assume $\bigcap_{j=1}^{n} \mathcal{O}_j \neq \emptyset$. Let $p \in \bigcap_{j=1}^{n} \mathcal{O}_j$. Then $p \in \mathcal{O}_j$ for $\forall j = 1, 2, \ldots, n$. Since each $\mathcal{O}_j$ is open, $\exists \epsilon_j > 0$ such that $N_{\epsilon_j}(p) \subseteq \mathcal{O}_j$. Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_j\} > 0$, then $N_\epsilon(p) \subseteq N_{\epsilon_j}(p) \subseteq \mathcal{O}_j$ for all j. Therefore, $N_\epsilon(p) \subseteq \bigcap_{j=1}^{n} \mathcal{O}_j$, and p is an interior point. ■

Corollary

Let (X, d) be a metric space.

1. If $\{\mathcal{C}_1, \cdots, \mathcal{C}_n\}$ is a finite collection of closed sets of X, then $\bigcup_{j=1}^{n} \mathcal{C}_j$ is closed. That is, a finite union of closed sets is closed.
2. If $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ is a collection of closed sets of X, then $\bigcap_{\alpha \in A} \mathcal{C}_\alpha$ is closed. That is, an arbitrary intersection of closed sets is closed.

Proof. This is immediate from De Morgan's laws. ■

3 Compact Sets

Now, we state the general definition of compact sets. We abbreviate the notation (X, d) and use X for metric spaces. This still implies that d is the distance function.

Definition 3.1: Open Cover

Let X be a metric space, and $E \subseteq X$. A collection $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ of open subsets of X is an **open cover** of E if

$$E \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha.$$

Definition 3.2: Compact Set

Let X be a metric space. A set $K \subseteq X$ is **compact** if every open cover of K has a finite subcover of K.

That is, if $\{\mathcal{O}_\alpha\}$ is an open cover of K, K is compact if $\exists\, \alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} \mathcal{O}_{\alpha_j}.$$

Example 3

Every finite set is compact.

Example 4

$I = (0, 1)$ is not compact. Consider $\mathcal{O}_n = (0, \frac{n}{n+1})$ for $n \in \mathbb{N}$.. Then, $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ is an open cover of I. Indeed, if $x \in I$, then $\exists n_0 \in \mathbb{N}$ such that $\frac{1}{n_0+1} < 1 - x$ by the Archedian property. Thus,

$$x \in \mathcal{O}_{n_0} \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n.$$

But, no finite subcover can cover I. Assume to the contrary that a finite subcover $\{\mathcal{O}_{n_1}, \mathcal{O}_{n_2}, \ldots, \mathcal{O}_{n_k}\}$ covers I. Let $N = \max\{n_1, \cdots, n_k\}$, then we have

$$(0, 1) \subseteq \bigcup_{j=1}^{k} \mathcal{O}_{n_j} = \left(0, \frac{N}{N+1}\right),$$

which gives a contradiction.

4 Properties of Compact Sets

Proofs in the section are done in a general metric space unless it mentions that the metric space is the real line.

Theorem 4.1: Heine-Borel

Every closed and bounded interval $[a, b] \subseteq \mathbb{R}$ is compact.

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$. Define

$$E = \{r \in [a, b], [a, r] \text{ is covered by a finite subcover of } \mathcal{U}\}.$$

Clearly, E is nonempty (since $a \in E$) and bounded. Thus $\exists \gamma = \sup E$ in $\mathbb{R}$ by the least upper bound property.

Claim. $\gamma = b$.

Suppose that $\gamma < b$. We will find a contradiction by constructing $s \in E$ such that $\gamma < s$. Since $\gamma \in U_\alpha$ for some open set $U_\alpha \in \mathcal{U}$, $\exists \epsilon > 0$ such that $N_\epsilon(\gamma) = (\gamma - \epsilon, \gamma + \epsilon) \subseteq U_\alpha$. Since $\gamma - \epsilon$ is not an upper bound of E, $\exists t \in E$ such that $\gamma - \epsilon < t < \gamma$. Thus, $[0, t]$ is covered by finitely many sets

$$U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}.$$

Now, choose any $s \in (\gamma, \gamma + \epsilon)$ such that $s < b$. Then,

$$[a, s] \subseteq \left(\bigcup_{j=1}^{n} U_{\alpha_j} \right) \cup U_\alpha,$$

i.e. $s \in E$. Also since $\gamma \in E$, so this completes the proof. ■

Note that if $X = \mathbb{R}$, closed and bounded is equivalent to compact (so a compact set is also closed and bounded). In a general metric space, every closed and bounded set is compact, but not every compact set is closed and bounded.

Then, is this definition equivalent to the sequential definition for $X = \mathbb{R}$? Yes!

Theorem 4.2

Let $K \subseteq \mathbb{R}$. Then K is compact if and only if every sequence in K has a subsequence that converges to a point in K.

Proof. ($\Rightarrow$) Let $\{p_n\}$ be a sequence in K, and let $E = \{p_n : n = 1, 2, \cdots\}$. If E is finite, then there exists a point $p \in E$ and a sequence $\{n_k\}$ with $n_1 < n_2 < \cdots$ such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The sequence $\{p_{n_k}\}$ obviously converges to $p \in K$.

Let E be infinite. Since K is closed, E has a limit point $p \in K$. Choose n_1 such that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{k-1}$, choose an integer $n_k > n_{k-1}$ so that

$$d(p, p_{n_k}) < \frac{1}{k}.$$

Such an integer n_k exists since every neighborhood of p contains infinitely many points of E. The sequence $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$ converging to $p \in K$.

($\Leftarrow$) Let p be a limit point of K. Then there exists a sequence of distinct points in K converging to p. Since each of its subsequence converge to p, hence $p \in K$, so K is closed.

Assume K is not bounded. For each $k \in \mathbb{N}$, choose a point $p_k \in K$ such that $d(p_k, p_0) \geq k$ for some fixed $p_0 \in X$. Then the sequence $\{p_k\}$ satisfies $d(p_k, p_0) \to \infty$, so it cannot have any convergent subsequence (since convergent sequences are bounded), a contradiction.

Since K is closed and bounded, it is compact. ■

Proving closedness and boundedness were enough to show that a set is compact, but this only applies when the metric space X is the real line. For general metric space, Heine-Borel does not apply.

Example 5

Let $X = \ell^\infty$ be the space of all bounded real sequences. That is,

$$X = \ell^\infty = \{\{x_n\}_{n=1}^\infty : \sup |x_n| < \infty\}$$

with the metric $d(\{x_n\}, \{y_n\}) = \sup |x_n - y_n|$. Now, consider the set

$$A = \{e_n : n \in \mathbb{N}\}$$

with $e_n = (0, \ldots, 0, 1, 0, \ldots, 0)$ where only the nth term of the sequence is one, and the other terms are zero.

Then it is easy to see that A is both closed and bounded. However, A is not compact.

Note that for $m \neq n$, $d(e_m, e_n) = 1$. Consider $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ where $U_n = B(e_n, 1/2) = \{x \in X : d(x, e_n) < 1/2\}$. Then $e_n \in U_n$, and U_n contains exactly e_n. Thus $\mathcal{U}$ is an open cover of A, but A does not have a finite subcover (since A is infinite).

This shows that one needs more constraints to imply that a set if compact in a general metric space.

Theorem 4.3

Let X be a metric space. If $K \subseteq X$ is compact, then

1. K is closed
2. If $F \subseteq K$ and F is closed, then F is compact.

Proof. (1) It is enough to show that K^{C} is open. Let $p \in K^{\mathsf{C}}$. For each $q \in K$, Let $\epsilon_q = d(p,q)/2$. Then, $N_{\epsilon_q}(p) \cap N_{\epsilon_q}(q) = \emptyset$. Since $\{N_{\epsilon_q}(q)\}_{q\in K}$ is an open cover of K, there exists q_1, q_2, ..., q_n such that

$$K \subseteq \bigcup_{j=1}^{n} N_{\epsilon_{q_j}}(q_j).$$

Let $\epsilon = \min\{q_1, \cdots, q_n\}$. Then, $N_\epsilon(p)$ does not intersect with $N_{\epsilon_{q_j}}(q_j)$ for all $j = 1, \ldots, n$. Thus, $N_\epsilon(p) \subseteq K^{\mathsf{C}}$, which proves that K is closed.

(2) Let $\{\mathcal{O}_\alpha\}_{\alpha\in A}$ be an open cover of F. Then,

$$\{\mathcal{O}_\alpha\}_{\alpha\in A} \cup F^{\mathsf{C}}$$

is an open cover of K. Since K is compact, there is a finite subcollection of $\{\mathcal{O}_\alpha\}_{\alpha\in A} \cup F^{\mathsf{C}}$ containing K, which also contains of F. ■

Corollary

If F is closed and K is compact, then $F \cap K$ is compact.

The previous theorem gives a simple proof for generalized Cantor's intersection property.

Corollary : Cantor's Intersection Property

If $K_1 \supseteq K_2 \supseteq \ldots$ is a nested family of nonempty compact sets, then their intersection $K = \bigcap_{n=1}^{\infty} K_n$ is nonempty and compact.

Proof. Since K is a closed subset of a compact set K_n, it is compact. To prove that K is nonempty, define a sequence $\{x_n\}$ with $x_i \in K_i$. Then $\{x_n\} \subseteq K$. If x_n converges to x, then since K is closed, $x \in K$. This shows that K is nonempty. ■

Theorem 4.4

The continuous image of a compact set is compact.

Proof. Suppose $K \in X$ is compact, and $f : K \to f(K)$ is continuous. Let $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$. Then $f(K) \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. Since the preimage of an open set is open, we have

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(\mathcal{O}_\alpha),$$

which gives that $\{f^{-1}(\mathcal{O}_\alpha)\}_{\alpha \in A}$ is an open cover of K. Since K is compact, there is a finite subcover of this open cover, i.e. there is $f^{-1}(\mathcal{O}_{\alpha_1})$, $f^{-1}(\mathcal{O}_{\alpha_2})$, ..., $f^{-1}(\mathcal{O}_{\alpha_n})$ such that

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(\mathcal{O}_{\alpha_i}).$$

Therefore

$$f(K) \subseteq \bigcup_{i=1}^{n} \mathcal{O}_{\alpha_i},$$

so $f(K)$ is compact. ■

Remark.

If $X = \mathbb{R}$, then the continuous image of a closed set need not be closed.

Remark.

If $X = \mathbb{R}$, then the continuous image of a bounded set need not be bounded.

This gives the Extreme value theorem.

Corollary : Extreme Value Theorem

Let $K \subseteq \mathbb{R}$ be a compact set. If $f : K \to \mathbb{R}$, then $f(x)$ attains its minimum and maximum value on K.

5 References

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