AMC 12A 2023 Notes

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November 8, 2023

The AMC (American Mathematics Competitions) held by the MAA (Mathematical Association of America) is the annual most famous competition. These are my solutions to the actual test when I took it. The link to the competition is here: (enter link when published). I thank Ian Baek for his solutions to questions 10 and 23.

Problem 1

Cities A and B are 45 miles apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 miles per hour. Leaving at the same time, Beth bikes toward A at 12 miles per hour. How many miles from City A will they be when they meet?

(A) 20 (B) 24 (C) 25 (D) 26 (E) 27

Solution Distance ratio is $18 : 12 = 3 : 2$, so $45 \cdot \dfrac{3}{3+2} = 27$. E

Problem 2

The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

(A) $1\frac{4}{5}$ (B) 2 (C) $2\frac{2}{5}$ (D) 3 (E) $3\frac{3}{5}$

Solution We have

$$\frac{1}{3}P + \frac{7}{2}O = \frac{3}{4}P + \frac{1}{2}O$$

where P is the weight of the pizza and O is the weight of the orange. We then get $\frac{5}{12}P = 3O = \frac{3}{4}$ and therefore $P = \frac{9}{5}$. A

Problem 3

How many positive perfect squares less than 2023 are divisible by 5?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution Observe that $2023 + 2 = 2025 = 45^2$. Therefore there are 5^2, 10^2, $\ldots$, 40^2. A

Problem 4

How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Solution The prime decomposition of the number is $2^{15} \cdot 3^5 \cdot 5^{15}$, so it is $243 \cdot 10^{15}$, which is $3 + 15 = 18$ digits. E

Problem 5

Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal 3?

(A) $\frac{2}{9}$ (B) $\frac{49}{216}$ (C) $\frac{25}{108}$ (D) $\frac{17}{72}$ (E) $\frac{13}{54}$

Solution The cases are $\{3, *, *, *\}$, $\{1, 2, *, *\}$, $\{2, 1, *, *\}$, and $\{1, 1, 1, *\}$. Therefore, the probability is

$$\frac{1}{6} + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^3 = \frac{49}{216}.$$ B

Problem 6

Points A and B lie on the graph of $y = \log_2 x$. The midpoint of $\overline{AB}$ is $(6, 2)$. What is the positive difference between the x-coordinates of A and B?

(A) $2\sqrt{11}$ (B) $4\sqrt{3}$ (C) 8 (D) $4\sqrt{5}$ (E) 9

Solution Let $A(a, \log_2 a)$ and $B(b, \log_2 b)$. Then, we have $a + b = 12$ and $ab = 16$ because $\log_2 a + \log_2 b = \log_2 ab = 4$. Therefore, $(a - b)^2 = (a + b)^2 - 4ab = 80$ and $a - b = 4\sqrt{5}$. D

Problem 7

A digital display shows the current date as in 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as 20230428. For how many dates in 2023 does each digit appear an even number of times in the 8-digit display for that date?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Solution 2023 already has an even number of 2 but odd number of 0 and 3, so there should be one more of 0 and 3. For the other two digits, we may have both 1, both 2. Casework gives 20230113, 20230131, 20230311, 20231013, 20231031, 20231103, 20231130, 20230223, 20230232, so there are 9 cases. E

Problem 8

Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution Let the sum of her quiz scores be s and the number n. We need to find $\frac{s}{n}$. We have

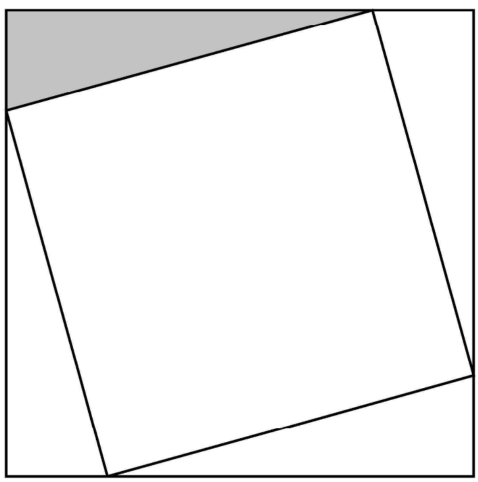
$$\frac{s+11}{n+1} = \frac{s}{n} + 1 \text{ and}$$
$$\frac{s+33}{n+2} = \frac{s}{n} + 2.$$

Solving the system gives $s = 21$ and $n = 3$, so $\frac{s}{n} = 7$. D

Problem 9

A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $2-\sqrt{3}$ (D) $\sqrt{3}-\sqrt{2}$ (E) $\sqrt{2}-1$



Solution Let a and b be the two sides of the triangle which are not the hypotenuse. Then, we have $4 \cdot \frac{1}{2}ab = 1$ and $a^2 + b^2 = 4$, so

$$a + b = \sqrt{3} \text{ and } ab = \frac{1}{2},$$

which gives

$$a = \frac{\sqrt{3}+1}{2} \text{ and } b = \frac{\sqrt{3}-1}{2}.$$

Therefore, $\frac{b}{a} = 2 - \sqrt{3}$. C

Problem 10

Positive real numbers x and y satisfy $y^3 = x^2$ and $(y-x)^2 = 4y^2$. What is $x + y$?

(A) 12 (B) 18 (C) 24 (D) 36 (E) 42

Solution (Guessing, my solution) Since the two equations are in polynomial form, we guess that x and y are integers, or at least rationals. Trying $(\pm 8, 4)$ and $(\pm 27, 9)$, we get that $(27, 9)$ is a solution. Therefore $x + y = 36$. D

Solution (Ian's solution) Since $(y-x)^2 = 4y^2$, $x^2 - 2xy - 3y^2 = (x-3y)(x+y) = 0$, and therefore $x = 3y$ because x and y are positive. We now have $y^3 = x^2 = 9y^2$, so $y = 3$, and $x = 27$. Therefore $x + y = 36$. D

Problem 11

What is the degree measure of the acute angle formed by lines with sloped 2 and $\frac{1}{3}$?

(A) 30 (B) 37.5 (C) 45 (D) 52.5 (E) 60

Solution Let $\tan\alpha = 2$ and $\tan\beta = \frac{1}{3}$. Then,

$$\tan(\alpha - \beta) = \frac{2 + \frac{1}{3}}{1 - 2 \cdot \frac{1}{3}} = 1,$$

so $\alpha - \beta = 45°$. C

Problem 12

What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

(A) 2023 (B) 2679 (C) 2941 (D) 3159 (E) 3235

Solution The desired value is

$$\begin{aligned}\sum_{n=1}^{9} (2n)^3 - (2n-1)^3 &= \sum_{n=1}^{9} 12n^2 - 6n + 1 \\ &= 2 \cdot 9(9+1)(2 \cdot 9 + 1) - 3 \cdot 9(9+1) + 9 = 3159. \text{ D}\end{aligned}$$

Problem 13

In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was 40% more than the number of games won by the right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

(A) 15 (B) 36 (C) 45 (D) 48 (E) 66

Solution (Guessing, my solution) If we let the number of right-handed players $2n$ and left-handed players n, the total number of players is $3n$, so it is a multiple of 3. The total number of games should be in the form $\frac{3n(3n-1)}{2}$, eliminating C and D. We also can let the number of games won by left-handed players $7k$ and the number of games won by right-handed players $5k$. Since there are no ties, the total number of games is $12k$, which is a multiple of 12. This gives B the answer because only 36 is a multiple of 12 out of the left answer choices. B

Problem 14

How many complex numbers satisfy the equation $z^5 = \overline{z}$, where $\overline{z}$ is the conjugate of the complex number z?

(A) 2 (B) 3 (C) 5 (D) 6 (E) 7

Solution Let $z = re^{i\theta}$. Then, we have

$$r^5 e^{i5\theta} = re^{-i\theta}.$$

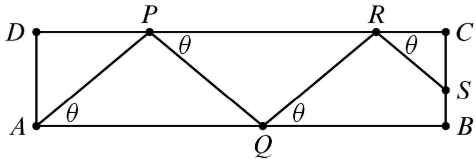
So either $r = 0$ or $r = 1$ and $e^{i5\theta} = e^{-i\theta}$. If $r = 0$, $z = 0$. If $r = 1$, we have $e^{i6\theta} = 1$, so z can be the complex 6th roots of unity. Therefore there are 1+6=7 such z. E

Problem 15

Usain is walking for exercise by zigzagging across a 100-meter by 30-meter rectangular field, beginning at point A and ending on the segment $\overline{BC}$. He wants to increase the distance walked by zigzagging as shown in the figure below ($APQRS$). What angle $\theta = \angle PAB = \angle QBC = \angle RQB = \cdots$ will produce a length that is 120 meters? (The figure is not drawn to scale. Do not assume that the zigzag path has exactly four segments as shown; it could be more or fewer.

(A) $\arccos\frac{5}{6}$ (B) $\arccos\frac{4}{5}$ (C) $\arccos\frac{3}{10}$ (D) $\arcsin\frac{4}{5}$ (E) $\arcsin\frac{5}{6}$

Solution Reflecting PQ over DC, APQ becomes a straight line. Moving QR and all the next segments to form a straight line makes a right triangle $\triangle ABS'$ where S' is the moved endpoint, $\angle SBA = 90°$, and $\angle SAB = \theta$. Then, $\cos\theta = \frac{100}{120} = \frac{5}{6}$, and $\theta = \arccos\frac{5}{6}$. A



Problem 16

Consider the set of complex numbers z satisfying $|1+z+z^2| = 4$. The maximum value of the imaginary part of z can be written in the form $\frac{\sqrt{m}}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Solution Since the norm of $1+z+z^2$ is 4, we have $1+z+z^2 = 4e^{i\theta}$ for some θ. Solving the quadratic equation, we get

$$z = \frac{-1+\sqrt{-3+16e^{i\theta}}}{2} = \frac{-1+\sqrt{-3+16\cos\theta+i16\sin\theta}}{2}.$$

Let $\sqrt{-3+16\cos\theta+i16\sin\theta} = a+bi$. We now want to maximize b. We have

$$a^2-b^2 = -3+16\cos\theta \text{ and } ab = 8\sin\theta.$$

If $\theta = \pi$, we get $ab = 0$ and $a^2-b^2 = -19$, which maximizes $b = \sqrt{19}$. Therefore, the maximum value of the imaginary part of z is $\frac{\sqrt{19}}{2}$, so $m+n = 21$. B

Problem 17

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ (B) $\frac{45}{1024}$ (C) $\frac{127}{1024}$ (D) $\frac{511}{1024}$ (E) $\frac{1}{2}$

Solution (Original solution) Since Flora can't go left, it will make 10 jumps in total. The probability is the number of the sequences of jumps Flora can make divided by 2^{10}. For the number of the sequences of jumps, Flora can either stop or skip 1, stop or skip 2, ..., stop or skip 9, which gives total 2^9 choices. Therefore, the probability is $\frac{2^9}{2^{10}} = \frac{1}{2}$. E

Solution (Guessing) Let p_n be the probability of Flora landing at n. Then, $p_1 = \frac{1}{2}$. For p_2, Flora can take either $\{2\}$ or $\{1,1\}$. Therefore,

$$p_2 = \frac{1}{2^2} + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

For p_3, Flora can take either $\{3\}$, $\{2,1\}$, $\{1,2\}$, or $\{1,1,1\}$. Then,

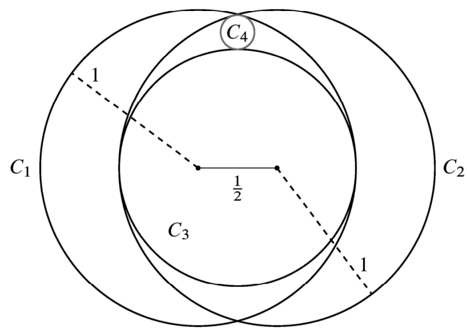
$$p_3 = \frac{1}{2^3} + \frac{1}{2^2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} + \left(\frac{1}{2}\right)^3 = \frac{1}{2}.$$

We can guess that $p_n = \frac{1}{2}$ for every n. E

Problem 18

Circles C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4?

(A) $\frac{1}{14}$ (B) $\frac{1}{12}$ (C) $\frac{1}{10}$ (D) $\frac{3}{28}$ (E) $\frac{1}{9}$



Solution Notice that the diameter of C_3 is the sum of the radii of C_1 and C_2 minus $\frac{1}{2}$, which is $\frac{3}{2}$. So the radius of C_3 is $\frac{3}{4}$. Let r be the radius of C_4. Connecting the centers of C_1, C_3, and C_4, we can construct a right triangle with lengths $\frac{1}{4}$, $\frac{3}{4}+r$, and hypotenuse $1-r$. Then, by the Pythagorean theorem, we have

$$\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}+r\right)^2 = (1-r)^2,$$

and therefore $r = \frac{3}{28}$. D

Problem 19

What is the product of all solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023?$$

(A) $(\log_{2023} 7 \cdot \log_{2023} 289)^2$ (B) $\log_{2023} 7 \cdot \log_{2023} 289$ (C) 1 (D) $\log_7 2023 \cdot \log_{289} 2023$ (E) $(\log_7 2023 \cdot \log_{289} 2023)^2$

Solution We change the equation to

$$\frac{\ln 2023}{\ln 7x} \cdot \frac{\ln 2023}{\ln 289x} = \frac{\ln 2023}{\ln 2023x}.$$

Dividing both sides by $\ln 2023$ and rearranging, we get

$$\ln 2023 \cdot \ln 2023x = \ln 7x \cdot \ln 289x,$$

which becomes

$$(\ln 7 + 2\ln 17)(\ln 7 + 2\ln 17 + \ln x) = (\ln 7 + \ln x)(2\ln 17 + \ln x).$$

This gives a quadratic equation of $\ln x$. Let $\ln x = a$, b. Then we have $x = e^a$, e^b, so the product of all solutions to the equation is the exponential of the sum of all $\ln x$. Since the coefficient of $\ln x$ term cancels out and becomes zero, the sum of all $\ln x$ is zero by Vieta, and the product of all the solutions is $e^0 = 1$. C

Problem 20

Rows 1, 2, 3, 4, and 5 of a triangular array of integers are shown below. Each row after the first row is formed by placing a 1 at each end of the row, and each interior entry is 1 greater than the sum of the two numbers diagonally above it in the previous row. What is the units digit of the sum of the 2023 numbers in the 2023rd row?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

1
1 1
1 3 1
1 5 5 1
1 7 11 7 1

Solution (Original solution) Let a_n be the sum of n numbers in the nth row. Then $a_{n+1} = 2a_n + n - 1$ because the sum of the numbers in the $n+1$th row except the two 1s are $2a_n + n - 1$. Then we get $a_n = 2^n - n$. Therefore,

$$2^{2023} - 2023 \equiv 8 - 3 = 5 \mod 10.$$ C

Solution (Guessing, my solution) Let a_n be the sum of n numbers in the nth row. We have $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 12$, $a_5 = 27$, and more. Notice that a_n has a recursion formula $a_{n+1} = 2a_n + n - 1$. Define the sequence $\{b_n\}$ by $b_n \equiv a_n \mod 10$. Then, computing b_n for several numbers, one can find that b_n is periodic with period 10. Therefore, $b_{2023} = b_3 = 5$. C

Problem 21

If A and B are vertices of a polyhedron, define the *distance* $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

(A) $\frac{7}{22}$ (B) $\frac{1}{3}$ (C) $\frac{3}{8}$ (D) $\frac{5}{12}$ (E) $\frac{1}{2}$

Solution (Counting) The total number of cases of randomly choosing three points Q, R, and S is $12 \cdot 11 \cdot 10 = 1320$. Since $1 \leq d(A, B) \leq 3$, we divide cases.

1. If $d(Q, R) = 3$
Notice that $d(R, S) < 3$ because there is a unique point Q such makes $d(Q, R) = 3$ for any R chosen. Therefore, if we choose any vertex R, which has 12 cases, Q is automatically chosen. S can be any other vertex, so there are 10 cases. The number of total possible cases is $12 \cdot 10 = 120$.

2. If $d(Q, R) = 2$ and $d(R, S) = 1$
First, choose any R. This has 12 cases. Then, to choose S, there should be an edge connecting R and S, so there are 5 possibilities to choose S. There are also 5 possibilities to choose Q to make $d(Q, R) = 2$. The number of total possible cases is $12 \cdot 5 \cdot 5 = 300$.

Therefore, the probability is

$$\frac{120 + 300}{1320} = \frac{7}{22}. \boxed{\text{A}}$$

Solution (Complementary Counting) We have either $d(Q, R) > d(R, S)$, $d(Q, R) = d(R, S)$, and $d(Q, R) < d(R, S)$. Without loss of generality, the probability that $d(Q, R) > d(R, S)$ and $d(Q, R) < d(R, S)$ are the same. We now focus on $d(Q, R) = d(R, S)$.

1. If $d(Q, R) = d(R, S) = 1$, then we have 12 choices of R and $5 \cdot 4$ choices of Q and S since it's a permutation. Therefore, we have total 240 choices.

2. If $d(Q, R) = d(R, S) = 2$, then we have 12 choices of R and $5 \cdot 4$ choices of Q and S since it's a permutation. Therefore, we have total 240 choices.

Therefore, the probability is

$$\frac{1}{2} \cdot \frac{1320 - (240 + 240)}{1320} = \frac{7}{22}. \boxed{\text{A}}$$

Problem 22

> Let f be the unique function defined on the positive integers such that
>
> $$\sum_{d|n} d \cdot f\left(\frac{n}{d}\right) = 1$$
>
> for all positive integers n, where the sum is taken over all positive divisors of n. What is $f(2023)$?
>
> (A) -1536 (B) 96 (C) 108 (D) 116 (E) 144

Solution Let $n = 2023$. Then we have

$$f(2023) + 7f(289) + 17f(119) + 119f(17) + 289f(7) + 2023f(1) = 1.$$

We will compute $f(1)$, $f(7)$, $f(17)$, $f(119)$, and $f(289)$ to find $f(2023)$.

Let $n = 1$. Then $f(1) = 1$.
Let $n = 7$. Then $f(7) + 7f(1) = 1$, so $f(7) = -6$.
Let $n = 17$. Then $f(17) + 17f(1) = 1$, so $f(17) = -16$.
Let $n = 119$. Then $f(119) + 7f(17) + 17f(7) + 119f(1) = 1$, so $f(119) = 96$.
Let $n = 289$. Then $f(289) + 17f(17) + 289f(1) = 1$, so $f(289) = -16$.

We now have

$$f(2023) - 7 \cdot 16 + 17 \cdot 96 - 119 \cdot 16 - 289 \cdot 6 + 2023 = 1,$$

so $f(2023) = 96$. B

> **Problem 23**
>
> How many ordered pairs of positive real numbers (a, b) satisfy the equation
>
> $$(1 + 2a)(2 + 2b)(2a + b) = 32ab?$$
>
> (A) 0 (B) 1 (C) 2 (D) 3 (E) an infinite number

Solution (Ian's solution) The equation is equivalent as $(1+2a)(1+b)(2a+b) = 16ab$. By AM-GM, we have

$$(1 + 2a)(1 + b)(2a + b) \geq 2\sqrt{2a} \cdot 2\sqrt{b} \cdot 2\sqrt{2ab} = 16ab.$$

Therefore, by the equality condition of AM-GM, we should have $1 = 2a$, $1 = b$, and $2a = b$. Therefore, $(a, b) = (\frac{1}{2}, 1)$, and this is the only pair. B

> **Problem 24**
>
> Let K be the number of sequences A_1, A_2, $\ldots$, A_n such that n is a positive integer less than or equal to 10, each A_i is a subset of $\{1, 2, 3, \ldots, 10\}$, and A_{i-1} is a subset of A_i for each i between 2 and n, inclusive. For example, $\{\}$, $\{5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$. What

is the remainder when K is divided by 10?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Problem 25

There is a unique sequence of integers $a_1, a_2, a_3, \ldots, a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x + \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

Solution (Guessing, my solution) Let $\tan x = a$. Then, we have

$$\begin{aligned}\tan 2x &= \frac{2a}{1-a^2} \\ \tan 3x &= \frac{a + \frac{2a}{1-a^2}}{1 - a \cdot \frac{2a}{1-a^2}} \\ &= \frac{3a - a^3}{1 - 3a^2}\end{aligned}$$

so $a_3 = -1$. Furthermore, we have

$$\begin{aligned}\tan 5x &= \frac{\frac{3a-a^3}{1-a^2} + \frac{2a}{1-a^2}}{1 - \frac{3a-a^3}{1-3a^2} \cdot \frac{2a}{1-a^2}} \\ &= \frac{5a - 10a^3 + a^5}{1 - 10a^2 + 3a^4}\end{aligned}$$

and

$$\begin{aligned}\tan 7x &= \frac{\frac{5a-10a^3+a^5}{1-10a^2+3a^4} + \frac{2a}{1-a^2}}{1 - \frac{5a-10a^3+a^5}{1-10a^2+3a^4} \cdot \frac{2a}{1-a^2}} \\ &= \frac{(1-a^2)(5a - 10a^3 + a^5) + 2a(1 - 10a^2 + 3a^4)}{(1 - 10a^2 + 3a^4)(1 - a^2) - 2a(5a - 10a^3 + a^5)}.\end{aligned}$$

Therefore $a_5 = 1$ and $a_7 = -1$. We can now guess that $a_n = 1$ if $n \equiv 1 \bmod 4$ and $a_n = -1$ if $n \equiv 3 \bmod 4$. Since $2023 \equiv 3 \bmod 4$, we can guess that $a_{2023} = -1$. C