AMC 12B 2023 Notes

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The AMC (American Mathematics Competitions) held by the MAA (Mathematical Association of America) is the annual most famous competition. These are my solutions to the actual test when I took it. The link to the competition is here. I thank Ian Baek for his solution to question 23.

Problem 1

Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

(A) $\frac{1}{12}$ (B) $\frac{1}{8}$ (C) $\frac{1}{6}$ (D) $\frac{2}{9}$ (E) $\frac{1}{4}$

Solution Total of $3 + \frac{1}{3} = \frac{10}{3}$ cups of juice is filled. Since each cup must have $\frac{1}{4} \cdot \frac{10}{3} = \frac{5}{6}$ full, we have $1 - \frac{5}{6} = \frac{1}{6}$. C

Problem 2

Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had $43. What is the original (before discount) price of the most expensive shoes he could afford to buy?

(A) $46 (B) $47 (C) $48 (D) $49 (E) $50

Solution Let the maximum price be x. Then, $x(80\%)(107.5\%) = x(0.8)(1.075) = 0.86x$. Since $0.86x \leq 43$, $x \leq 50$. E

Problem 3

A 3-4-5 right triangle is inscribed in circle A, and a 5-12-13 right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B?

(A) $\frac{1}{9}$ (B) $\frac{25}{169}$ (C) $\frac{4}{25}$ (D) $\frac{1}{5}$ (E) $\frac{9}{25}$

Solution If right triangle is inscribed in a circle, then the diameter of the circle is equal to the triangle's hypotenuse. Therefore, $r_a = \frac{5}{2}$, and $r_b = \frac{13}{2}$. Therefore the ratio is $(\frac{5}{13})^2 = \frac{25}{169}$. B

Problem 4

Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

(A) 162.5 (B) 1625 (C) 16250 (D) 162500 (E) 1625000

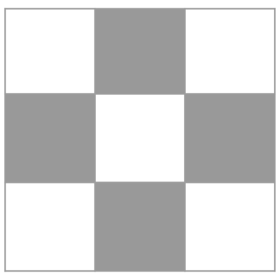
Solution Since 6.5 millimeters is 0.65 centimeters and 25 meters is 2500 centimeters, we have $0.65 \cdot 2500 = 1625$. B

Problem 5

You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 8

Solution With the following guess, the minimum is 4 squares. B



Problem 6

When the roots of the polynomial

$$P(x) = (x-1)(x-2)^2(x-3)^3 \cdot (x-10)^{10}$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Solution The degree of $P(x)$ is $1 + 2 + \cdots + 10 = 55$. When $x < 1$, $P(x)$ is a multiple of 55 negative numbers, so its negative. When $x = a_i$ is a root for some i, then $P(x)$ is both positive or both negative on (a_{i-1}, a_i) and (a_i, a_{i+1}) if the root a_i has even multiplicity, and is odd for one and even for the other if the root a_i has odd multiplicity. Therefore, $P(x)$ is positive in $(1, 2)$, $(2, 3)$, $(5, 6)$, $(6, 7)$, $(9, 10)$, and $(10, \infty)$, and there are six intervals that makes $P(x)$ positive. D

Problem 7

For how many integers n does the expression

$$\sqrt{\frac{\log(n^2) - (\log n)^2}{\log n - 3}}$$

represent a real number, where log denotes the base 10 logarithm?

(A) 2 (B) 3 (C) 900 (D) 901 (E) 902

Solution Let $\log n = x$. To make

$$\sqrt{\frac{2x - x^2}{x - 3}}$$

a real number, the value inside the square root must be greater or equal to 0. If $x > 3$, then $2x - x^2 \leq 0$, so there aren't such x. If $x < 3$, then $2x - x^2 \geq 0$, so either $x = 0$ or $2 \leq x < 3$ since a logarithm function cannot have negative values if n is an integer. Therefore, we have $n = 1$ or $100 \leq n < 1000$, so there are 901 such n. D

Problem 8

How many nonempty subsets B of $\{0, 1, 2, 3, \ldots, 12\}$ have the property that the number of elements in B is equal to the least element of B? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.

(A) 108 (B) 136 (C) 144 (D) 156 (E) 256

Solution We divide cases.
If the least element is 1, then we are done, and there is 1 case.
If the least element is 2, then the other element should be greater than 2, so there are 10 cases.
If the least element is 3, then the other 2 elements should be greater than 3, so there are $\binom{9}{2} = 36$ cases.
If the least element is 4, then the other 3 elements should be greater than 4, so there are $\binom{8}{3} = 56$ cases.
If the least element is 5, then the other 4 elements should be greater than 5, so there are $\binom{7}{4} = 35$ cases.
If the least element is 6, then the other 5 elements should be greater than 6, so

there are $\binom{6}{5} = 6$ cases.
If the least element is 7, there should be 6 more elements greater than 7, but there are only 5 elements greater than 7.
Therefore, the number of cases is $1 + 10 + 36 + 56 + 35 + 6 = 144$. C

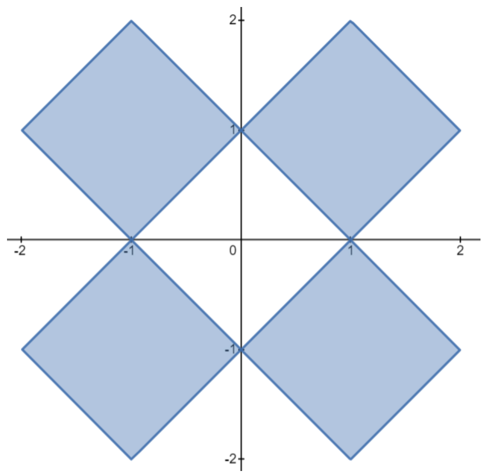
Problem 9

What is the area of the region in the coordinate defined by

$$\big||x| - 1\big| + \big||y| - 1\big| \leq 1?$$

(A) 2 (B) 4 (C) 8 (D) 12 (E) 15

Solution With the following diagram, the area is 8. Drawing the area when $x \geq 1$ and $y \geq 1$, and then reflecting over $(1, 1)$, and reflecting again over $(0, 0)$ will give you the diagram. C



Problem 10

In the xy-plane, a circle of radius 4 with center on the positive x-axis is tangent to the y-axis at the origin, and a circle of radius 10 with center on the positive y-axis is tangent to the x-axis at the origin. What is the slope of the line passing through the two points at which these circles intersect?

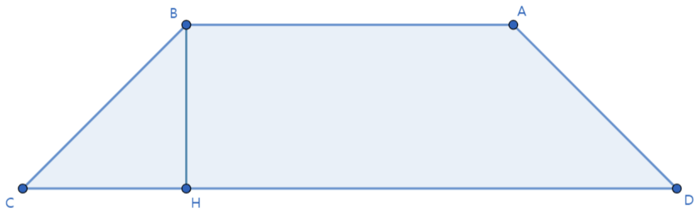
(A) $\frac{1}{\sqrt{29}}$ (B) $\frac{2}{7}$ (C) $\frac{2}{\sqrt{29}}$ (D) $\frac{2}{5}$ (E) $\frac{3}{7}$

Solution The equations of the two circles are $(x-4)^2+y^2 = 16$ and $x^2+(y-10)^2 = 100$. If we let the intersection point (x, y), then solving the system gives $8x = 20y$. Therefore, the slope is $\frac{y}{x} = \frac{8}{20} = \frac{2}{5}$. D

Problem 11

What is the maximum area of an isosceles trapezoid that has legs of length 1 and one base twice as long as the other?

(A) $\frac{8}{7}$ (B) $\frac{5}{4}$ (C) $\frac{3\sqrt{3}}{4}$ (D) $\frac{3}{2}$ (E) $\frac{5\sqrt{2}}{4}$



Solution Let the top base k and bottom base $2k$. If H is the perpendicular feet from B to $\overline{CD}$, then since $\overline{CH} = \frac{k}{2}$, the height is $\overline{BH} = \sqrt{1 - \frac{k^2}{4}}$. The maximum area is

$$\begin{aligned} S &= \frac{1}{2}(k+2k)\sqrt{1-\frac{k^2}{4}} \\ &= \frac{3}{2}\sqrt{k^2 - \frac{k^4}{4}} \\ &= \frac{3}{2}\sqrt{-\left(\frac{k^2}{2}-1\right)^2+1} \\ &\geq \frac{3}{2}\sqrt{1} = \frac{3}{2}. \quad \boxed{\text{D}} \end{aligned}$$

Problem 12

For complex numbers $u = a + bi$ and $v = c + di$ (where $i = \sqrt{-1}$), define the binary operation

$$u \otimes v = ac + bdi.$$

Suppose z is a complex number such that $z \otimes z = z^2 + 40$. What is $|z|$?

(A) 2 (B) $\sqrt{5}$ (C) $\sqrt{10}$ (D) 5 (E) $5\sqrt{2}$

Solution Let $z = a + bi$. Then, $z \otimes z = a^2 + b^2 i$, and $z^2 + 40 = a^2 - b^2 + 40 + 2abi$. Therefore, $a^2 = a^2 - b^2 + 40$, and $b^2 = 2ab$. Solving the system gives $b = 2\sqrt{10}$ and $a = \sqrt{10}$. Therefore, $|z| = \sqrt{a^2 + b^2} = 5\sqrt{2}$. $\boxed{\text{E}}$

Problem 13

A rectangular box $\mathcal{P}$ has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of $\mathcal{P}$ is 13, the sum of the areas of all 6 faces of $\mathcal{P}$ is $\frac{11}{2}$, and the volume of $\mathcal{P}$ is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of $\mathcal{P}$?

(A) $\dfrac{3}{8}$ (B) $\dfrac{9}{8}$ (C) $\dfrac{3}{2}$ (D) 2 (E) $\dfrac{9}{4}$

Solution We have $4(a+b+c) = 13$, $2(ab+bc+ca) = \frac{11}{2}$, and $abc = \frac{1}{2}$. The longest diagonal has length $\sqrt{a^2+b^2+c^2}$. Therefore, the length of the diagonal is

$$\begin{aligned}\sqrt{a^2+b^2+c^2} &= \sqrt{(a+b+c)^2 - 2(ab+bc+ca)} \\ &= \sqrt{\left(\frac{13}{4}\right)^2 - \frac{11}{2}} \\ &= \sqrt{\frac{81}{16}} = \frac{9}{4}. \quad \boxed{\text{E}}\end{aligned}$$

Problem 14

For how many ordered pairs of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution Notice that the pair (a, b) is uniquely determined if the three roots are given. We now look at the roots. Let α, β, and γ be the three roots of the polynomial. Then, by Vieta, $\alpha\beta\gamma = -6$. Therefore, the roots are $(\alpha, \beta, \gamma) = (-1, 2, 3), (1, -2, 3), (1, 2, -3), (-1, -2, -3)$, and $(1, -1, 6)$. There are 5 cases. $\boxed{\text{B}}$

Problem 15

Suppose that a, b, and c are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1$.

II. If $\gcd(c, 210) = 1$, then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.

III. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = 1$ and $\gcd(b, 15) = 1$.

(A) I only (B) III only (C) I and II only (D) II and III only (E) I,II, and III

Solution Multiplying both sides by 210, we get

$$15a + 14b = c.$$

Let $\gcd(a, 14) = d > 1$. Then, $d \mid \gcd(c, 14) = \gcd(15a + 14b)$, so c and 14 have a common factor. Therefore, c and 210 also have a common factor, so I is false. This leaves (D) and (E). Notice that if III is true, then II is also true. There are no possibilities of having III true and II false, so the answer is D.

Problem 16

In the state of Coinland, coins have values of 6, 10, and 15 cents. Suppose x is the value in cents of the most expensive item in Coinland that cannot be purchased using these coins with exact change. What is the sum of the digits of x?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Solution In the diagram below, the price of the item that can be purchased with exact change is colored gray. The desired value is 29, so $2 + 9 = 11$. E

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42

Problem 17

The three side lengths of a triangle are in arithmetic progression with shortest side of length 6. One of the interior angles measures 120°. What is the area of the triangle?

(A) $8\sqrt{6}$ (B) $14\sqrt{2}$ (C) $12\sqrt{3}$ (D) $15\sqrt{3}$ (E) $20\sqrt{2}$

Solution Let the three sides have length 6, $6 + d$, $6 + 2d$ where $d > 0$. Then, the longest side has length $6 + 2d$, and the opposite angle will measure 120°. Then, by the second law of cosines,

$$(6 + 2d)^2 = 6^2 + (6 + d)^2 - 2 \cdot 6(6 + d) \cos 120^\circ.$$

Rearranging the equation, we have $3d^2 + 6d - 72 = 0$, and $d^2 + 2d - 24 = 0$. Therefore, $d = 4$. The area of the triangle is

$$\frac{1}{2} \cdot 6(6+4)\sin 120^\circ = \frac{1}{2} \cdot 60 \cdot \frac{\sqrt{3}}{2} = 15\sqrt{3}. \quad \boxed{\text{D}}$$

Problem 18

Last academic year Yolanda and Zelda took different courses that did not necessarily administer the same number of quizzes during each of the two semesters. Yolanda's average on all the quizzes she took during the first semester was 3 points higher than Zelda's average on all the quizzes she took during the first semester. Yolanda's average on all the quizzes she took during the second semester was 18 points higher than her average for the first semester and was again 3 points higher than Zelda's average on all the quizzes Zelta took during her second semester. Which one of the following statements cannot possibly be true?

(A) Yolanda's quiz average for the academic year was 3 points higher than Zelda's.

(B) Yolanda's quiz average for the academic year was 22 points higher than Zelda's.

(C) Zelda's quiz average for the academic year was higher than Yolanda's.

(D) Zelda's quiz average for the academic year equaled Yolanda's.

(E) If Zelda had scored 3 points higher on each quiz she took, then she would have had the same average for the academic year as Yolanda.

Solution We eliminate the other answers. If Yolanda and Zelda took the same number of quizzes, (A) and (E) are possible scenarios and, thus can be eliminated. If Yolanda took few quizzes (like 1) and Zelta took many quizzes during the second semester, then it is possible to have Zelda's year average equal or higher by a small value than Yolanda's even though Zelda's semester average is lower because the quiz scores are generally higher in the second semester. Therefore, (C) and (D) could be eliminated, making (B) the answer. $\boxed{\text{B}}$

Problem 19

Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

(A) $\frac{1}{4}$ (B) $\frac{3}{10}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

Solution Let each bin have a, b, and c balls. Then, the total number of cases is the number of 3-tuples (a, b, c) such that $a + b + c = 2023$. This is equal to $\binom{2023+2}{2}$. If each bin has an odd number of balls, we can let $a = 2k + 1$, $b = 2l + 1$, and

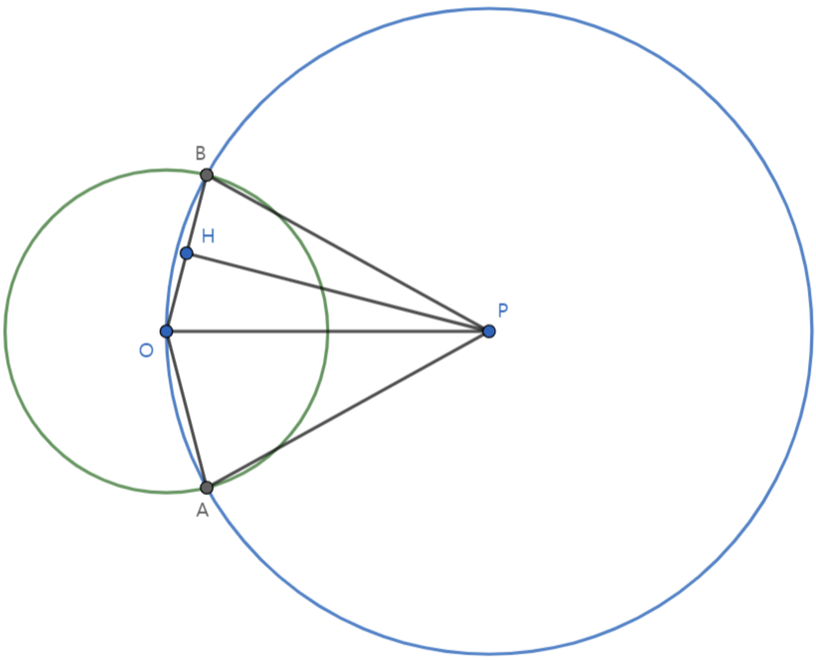
$c = 2m + 1$. Then, the number of cases is the number of 3-tuples (k, l, m) such that $k + l + m = 1010$. This is equal to $\binom{1010+2}{2}$. Therefore,

$$\frac{\binom{1012}{2}}{\binom{2025}{2}} = \frac{1012 \cdot 1011}{2025 \cdot 2024} = \frac{1011}{4050} \approx \frac{1}{4}. \quad \boxed{\text{A}}$$

Problem 20

Cyrus the frog sits on a flat surface. He jumps, landing 2 feet away. He then chooses a direction at random and again jumps 2 feet. What is the probability that after the second jump Cyrus lands within 1 foot of his starting position?

(A) $\dfrac{\arctan \frac{1}{2}}{\pi}$ (B) $\dfrac{2 \arcsin \frac{1}{4}}{\pi}$ (C) $\dfrac{1}{6}$ (D) $\dfrac{1}{5}$ (E) $\dfrac{\sqrt{3}}{8}$



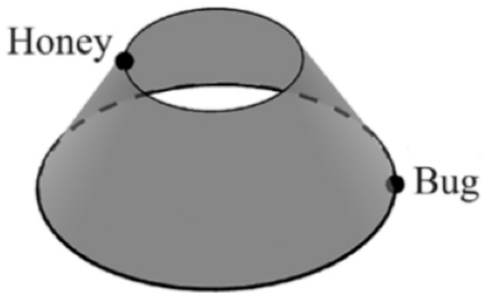
Solution Let O be the starting position and P be where Cyrus landed. If he jumps again, he will land somewhere on the blue circle. We want Cyrus to land on the minor arc AOB. Let the perpendicular foot from P to $\overline{BO}$ be H, and let $\angle BPH = \theta$. Then, the desired probability is $\frac{4\theta}{2\pi}$. Since $\overline{BO} = 1$ and $\overline{BH} = \frac{1}{2}$, $\sin\theta = \frac{1}{4}$. Therefore, the probability is

$$\frac{4\theta}{2\pi} = \frac{4 \arcsin \frac{1}{4}}{2\pi} = \frac{2 \arcsin \frac{1}{4}}{\pi}. \quad \boxed{\text{B}}$$

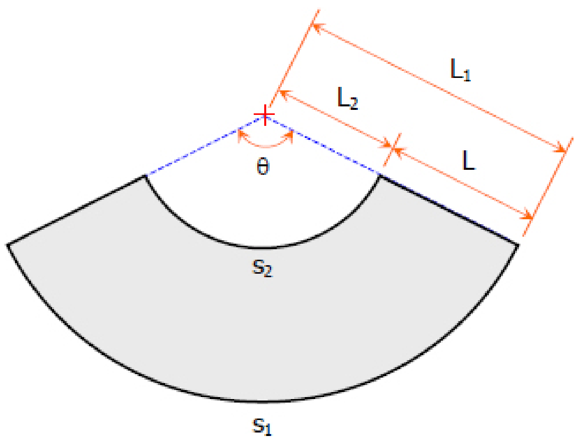
Problem 21

A lampshade is made in the form of the lateral surface of the frustum of a right circular cone. The height of the frustum is $3\sqrt{3}$ inches, its top diameter is 6 inches, and its bottom diameter is 12 inches. A bug is at the bottom of the lampshade and there is a glob of honey on the top edge of the lampshade at the spot farthest from the bug. The bug wants to crawl to the honey, but it must stay on the surface of the lampshade. What is the length in inches of its shortest path to the honey?

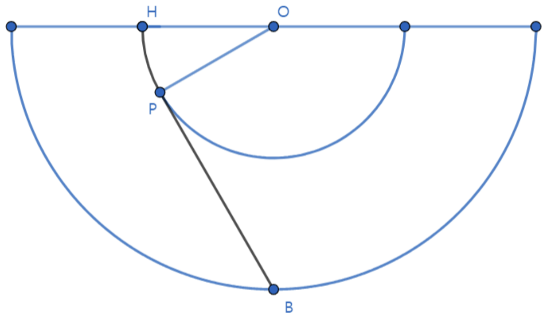
(A) $6\sqrt{3}$ (B) $6\sqrt{5}$ (C) $6\sqrt{3}+\pi$ (D) $6+3\pi$ (E) $6+6\pi$



Solution Extending the surface upward to make a circular cone. Then, the planar figure becomes like the figure below. By the Pythagorean theorem on the cross-



section, we have $L_1 = 12$, $L = L_2 = 6$. Since S_1 is the circumference of the bottom circle, $S_1 = 24\pi$. Therefore, $2L_1 \cdot \theta = 24\pi$, so $\theta = \pi$.



In the planar figure, let B be the point where the bug is, H the point where the honey is, and O the center of the top and bottom circles. To find the shortest path from B to H, the bug should move along the line that passes B and is tangent to the smaller circle until P, the tangency point. From P, the bug should move along the arc POH to H. In $\triangle PBO$, since $\angle BPO = 90°$, $\overline{OB} = 12$ and $\overline{OP} = 6$, $\overline{PB} = 6\sqrt{3}$ and $\angle POB = 60°$. Then, we have $\angle POH = 30°$, and therefore the length of arc POH is $12\pi \cdot \frac{30}{360} = \pi$. This gives the length of the shortest path, which is $6\sqrt{3} + \pi$. C

Problem 22

A real-valued function f has the property that for all real numbers a and b,

$$f(a+b) + f(a-b) = 2f(a)f(b).$$

Which one of the following *cannot* be the value of $f(1)$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Solution Substituting $b = 0$, we have $2f(a) = 2f(a)f(0)$. Therefore, $f(a) = 0$ for all a, or $f(0) = 1$. If $f(0) = 1$, then substituting $a = b = \frac{1}{2}$ gives $f(1)+1 = 2f(\frac{1}{2})^2$. However, since

$$f(1) = 2\left(f\left(\frac{1}{2}\right)\right)^2 - 1 \geq -1,$$

$f(1)$ can never be -2. A

Problem 23

When a standard 6-sided die is rolled n times, the product of the n numbers rolled can be any of 936 possible values. What is n?

(A) 6 (B) 8 (C) 9 (D) 10 (E) 11

Solution Let the product of the n numbers be a. We want to find n such that makes 936 different values of a. We divide cases.

If $v_5(a) = n$, then $a = 5^n$, so there is 1 case.

If $v_5(a) = n-1$, then the other number is either a divisor of $6 = 2 \cdot 3$ or $4 = 2^2 \cdot 3^0$, so there are $(1+1)(1+1)+1 = 5$ cases.

If $v_5(a) = n-2$, then the other two numbers is either a divisor of $6^2 = 2^2 \cdot 3^2$, a divisor of 8 but not 16 ($2^3 \cdot 3^0$ and $2^3 \cdot 3^1$), or a divisor of 16 ($2^4 \cdot 3^0$). There are $(2+1)(2+1)+2+1 = 12$ cases.

To generalize, if $v_5(a) = n-k$, then the other k numbers is either a divisor of $6^k = 2^k \cdot 3^k$, a divisor of 2^{k+1} but not 2^{k+2} (k cases), and so on. There are $(k+1)^2 + k + (k-1) + \cdots + 1 = \frac{(3k+2)(k+1)}{2}$ cases.

The total cases are

$$\begin{aligned}
\sum_{k=0}^{n} \frac{3k^2+5k+2}{2} &= \frac{1}{2}\sum_{k=0}^{n} 3k^2+5k+2 \\
&= \frac{1}{2}\left(2 + \frac{n(n+1)(2n+1)}{2} + \frac{5n(n+1)}{2} + 2n\right) \\
&= \frac{1}{2}(n^3+4n^2+5n+2) = 936.
\end{aligned}$$

Therefore, $n^3 + 4n^2 + 5n - 1870 = 0$, and $n = 11$. $\boxed{\text{E}}$

Problem 24

Suppose that a, b, c, and d are positive integers satisfying all of the following relations.

$$\begin{aligned}
abcd &= 2^6 \cdot 3^9 \cdot 5^7 \\
\operatorname{lcm}(a,b) &= 2^3 \cdot 3^2 \cdot 5^3 \\
\operatorname{lcm}(a,c) &= 2^3 \cdot 3^3 \cdot 5^3 \\
\operatorname{lcm}(a,d) &= 2^3 \cdot 3^3 \cdot 5^3 \\
\operatorname{lcm}(b,c) &= 2^1 \cdot 3^3 \cdot 5^2 \\
\operatorname{lcm}(b,d) &= 2^2 \cdot 3^3 \cdot 5^2 \\
\operatorname{lcm}(c,d) &= 2^2 \cdot 3^3 \cdot 5^2
\end{aligned}$$

What is $\gcd(a,b,c,d)$?

(A) 3 (B) 6 (C) 15 (D) 30 (E) 45

Solution Let $a = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3}$, $b = 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3}$, $c = 2^{c_1} \cdot 3^{c_2} \cdot 5^{c_3}$, and $d = 2^{d_1} \cdot 3^{d_2} \cdot 5^{d_3}$.

Comparing the exponents for 2, we have $\max(a_1,b_1) = \max(a_1,c_1) = \max(a_1,d_1) = 3$, so $a_1 = 3$. Also, since $\max(b_1,d_1) = \max(c_1,d_1) = 2$, $d_1 = 2$. We are left with $b_1 + c_1 = 1$, so one of b_1 and c_1 is 1, and the other is 0. Therefore $\min(a_1,b_1,c_1,d_1) = 0$, and $v_2\big(\gcd(a,b,c,d)\big) = 0$.

Comparing the exponents for 3, we have $\max(a_2, d_2) = \max(b_2, d_2) = \max(c_2, d_2) = 3$, so $d_2 = 3$. Also, since $\max(a_2, c_2) = \max(b_2, c_2) = 3$, $c_2 = 3$. We are now left with $a_2 + b_2 = 3$. Since $\max(a_2, b_2) = 2$, one of a_2 and b_2 is 2, and the other is 1. Therefore $\min(a_2, b_2, c_2, d_2) = 1$, and $v_3\big(\gcd(a, b, c, d)\big) = 1$.

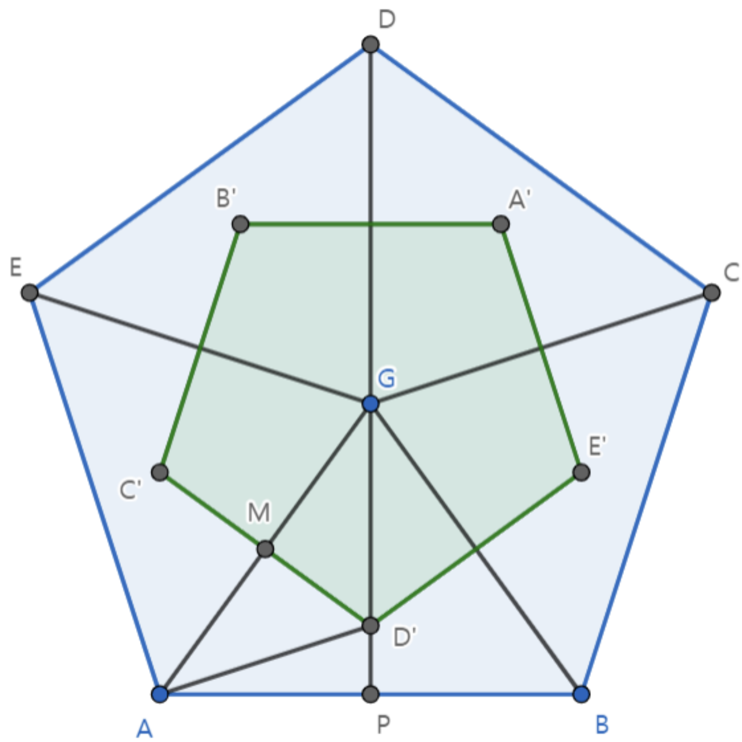
Comparing the exponents for 5, we have $\max(a_3, b_3) = \max(a_3, c_3) = \max(a_3, d_3) = 3$, so $a_3 = 3$. Also, since $\max(b_3, c_3) = \max(b_3, d_3) = 2$, $b_3 = 2$. We are now left with $c_3 + d_3 = 2$. Since $\max(c_3, d_3) = 2$, one of c_3 and d_3 is 2, and the other is 0. Therefore $\min(a_3, b_3, c_3, d_3) = 0$, and $v_5\big(\gcd(a, b, c, d)\big) = 0$.

Combining the results, we have $\gcd(a, b, c, d) = 2^0 \cdot 3^1 \cdot 5^0 = 3$. A

Problem 25

A regular pentagon with area $1 + \sqrt{5}$ is printed on paper and cut out. All five vertices are folded to the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

(A) $\sqrt{5} - 1$ (B) $8 - 3\sqrt{5}$ (C) $\dfrac{2 + \sqrt{5}}{3}$ (D) $\dfrac{1 + \sqrt{5}}{2}$ (E) $4 - \sqrt{5}$



Solution Let the regular pentagon be $ABCDE$, and G its center. The five vertices of the smaller pentagon is formed by the intersections of the perpendicular bisectors of $\overline{GA}$, $\overline{GB}$, $\overline{GC}$, $\overline{GD}$, and $\overline{GE}$. Let the smaller regular pentagon be $A'B'C'D'E'$. Then, G is also the center of the smaller pentagon. Therefore, the area ratio between the two pentagons are square of the ratio between the length of $\overline{GD'}$ and $\overline{GA}$.

Let M be the midpoint of $\overline{GA}$. Then,

$$\begin{aligned}\overline{GA} &= 2\,\overline{GM} \\ &= 2\,\overline{GD'}\cos\angle D'GM \\ &= 2\,\overline{GD'}\cos 36^\circ \\ &= \frac{1+\sqrt{5}}{2}\,\overline{GD'},\end{aligned}$$

and

$$\overline{GD'} = \frac{2}{1+\sqrt{5}}\,\overline{GA} = \frac{-1+\sqrt{5}}{2}\,\overline{GA}.$$

Finally, the area of the smaller pentagon is

$$(1+\sqrt{5})\left(\frac{-1+\sqrt{5}}{2}\right)^2 = \sqrt{5}-1. \quad \boxed{\text{A}}$$