# Complex Numbers

Joshua Im

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This handout covers complex numbers in intermediate-level. Since this handout is aimed for students preparing for computational competitions, techniques used in olympiads (e.g. complex bashing) are not included here.

This handout is written with assuming that people reading this are already familiar with the standard form of complex numbers (including the arithmetic of it). If you're not familiar, read this handout by TheCALT.

## 1 Definitions and Terminology

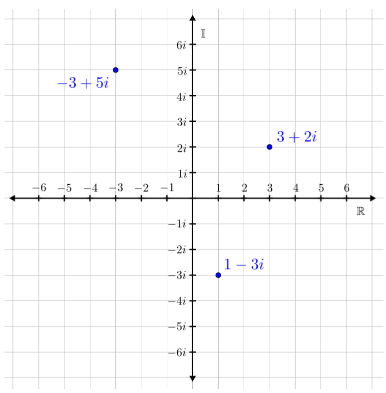
Recall the definition of a complex number.

**Definition 1: Complex Number (standard form)**

A **complex number** $z$ is a number of the form

$$z = a + bi$$

where $a$ and $b$ are real numbers, and $i = \sqrt{-1}$.



We wish to interpret these numbers geometrically. Recall that we plot real numbers in a line. We can do a similar thing for complex numbers. Let $x$-axis be the real axis, and $y$-axis be the imaginary axis. Then, for any complex number $a + bi$, there is exactly one point corresponding, namely $(a, b)$. The figure above shows $-3 + 5i$ at $(-3, 5)$, $3 + 2i$ at $(3, 2)$, and $1 - 3i$ at $(1, -3)$.

If we look at the number with polar coordinates, there exist a unique $(r, \theta)$ for the point. Thus, the parameters $r$ and $\theta$ give us an alternative way of specifying complex numbers.

**Definition 2: Complex Number (trigonometric form)**

A **complex number** $z$ is a number of the form

$$z = r(\cos\theta + i\sin\theta)$$

where $r$ is the *norm* of $z$, and $0 \leq \theta < 2\pi$.

Wait, what is a *norm*? Notice that when $z$ is on the coordinate plane, $r$ is the distance between the origin and $z$.

**Definition 3: Norm**

A norm of a complex number is

$$|z| = r = \sqrt{a^2 + b^2}.$$

We write the norm as an absolute value because the norm is the distance between $z$ and the origin. Now, With Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can write complex numbers in exponential form;

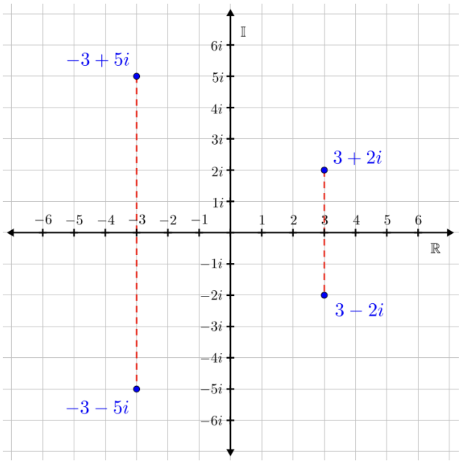
$$z = re^{i\theta}.$$

**Definition 4: Conjugate**

Let $z = a + bi$ be a complex number. Then the **conjugate** of $z$ is defined by

$$\bar{z} = a - bi.$$

It is not hard to show that $\bar{z} = re^{-i\theta}$. Also, a complex conjugate can also be thought of as the reflection of a complex number about the real axis in the complex plane. The figure shows the points corresponding to $-3 + 5i$, $3 + 2i$, and their complex conjugates.




## 2 Basic Operations

Readers should already be familiar with basic arithmetic and properties of complex numbers, in standard form. We move on to the operations in trigonometric form.

Let $z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$. Then,

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ &= r_1 r_2 \big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big). \end{aligned}$$

Division is defined similarly.

**Theorem 1: De Moivre's Theorem**

Let $\theta$ be an angle. Then,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This theorem makes computing complex powers much faster.

**Example 1**

Compute $(\sqrt{3}+i)^6$.

**Solution** We have

$$\begin{aligned}(\sqrt{3}+i)^6 &= 2^6 \cdot \left(\frac{\sqrt{3}+i}{2}\right)^6 \\ &= 2^6 \cdot (e^{i\pi/6})^6 \\ &= 2^6 \cdot e^{i\pi} = -64.\end{aligned}$$

**Example 2 (2005 AIME II P9)**

For how many positive integers $n$ less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real $t$?

**Walkthrough** The problem looks a lot like De Moivre's theorem.

1. Change the equation to De Moivre's theorem with $\cos t = \sin\left(\frac{\pi}{2}-t\right)$, and vise versa.
2. Derive $\cos\left(\frac{n\pi}{2}-nt\right) = \sin nt$, and vise versa.
3. Finish. The answer is 250.

## 3 Complex Roots

We first state a lemma about the relationship between conjugate and the norm.

**Lemma**

We have $|z|^2 = z\bar{z}$.

Recall when the discriminant of a quadratic equation (with real coefficients) is negative, then you get complex roots. These come in *pairs*: if $z$ is a root to an equation, then $\bar{z}$ is also a root.

**Lemma**

Let $P(x)$ be a polynomial. If a complex number $z$ is a root of $P(x)$, then its conjugate $\bar{z}$ is also a root.

**Example 3 (AMC 12B 2021 Spring P18)**

Let $z$ be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

**Walkthrough** Change $|z|^2 = z\bar{z}$ and bash. The answer is -2.

**Example 4 (AMC 12B 2021 Fall P21)**

For real numbers $x$, let

$$P(x) = 1 + \cos(x) + i\sin(x) - \cos(2x) - i\sin(2x) + \cos(3x) + i\sin(3x)$$

where $i = \sqrt{-1}$. For how many values of $x$ with $0 \le x < 2\pi$ does $P(x) = 0$?

**Walkthrough** Let $z = \cos x + i \sin x$.

1. Change $P(x)$ in terms of $z$.
2. What norm should $z$ have?
3. $P(x) = 0$ has one real root and two conjugate roots.
4. Let the real root be $k$. Prove that $-1 < k < 0$.
5. What value should $kz\bar{z}$ have?
6. Is it possible to have a root with norm 1?
7. Finish. The answer is 0.

## 4 Trigonometry

Let $z = e^{i\theta} = \cos\theta + i\sin\theta$. Then, we have $\bar{z} = e^{-i\theta} = \cos\theta - i\sin\theta$. This gives us the relationship with trigonometry.

**Theorem 2: Relationship with Trigonometry**

We have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

**Example 5**

Find $\cos 4\theta$ in terms of $\cos\theta$.

**Walkthrough** Let $2\cos\theta = e^{i\theta} + e^{-i\theta}$.

1. Square both sides, and rearrange the constant.
2. Divide both sides by 2. What do you get?
3. Repeat and finish. The answer is $8\cos^4\theta - 8\cos^2\theta + 1$.

**Example 6 (AMC 12A 2021 Spring P22)**

Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is $abc$?

**Walkthrough** But why $abc$.. too bash

1. Let $\omega = e^{2\pi i/7}$.
2. Write $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$ in terms of $\omega$.
3. Write $a$, $b$, and $c$ in terms of $\omega$.
4. Finish. The answer is $1/32$.

## 5 Roots of Unity

**Definition 5: Root of Unity**

The **nth roots of unity** are complex numbers $z$ satisfying the equation

$$z^n - 1 = 0.$$

By the fundamental theorem of algebra, there are $n$ complex roots to this equation. These numbers have a pattern.

**Theorem 3**

The **nth roots of unity** have the form

$$z = e^{2\pi i\cdot k/n},$$

where $0 \le k < n$.

To see why this is true, plug in the formula. This is a powerful tool to close the border between algebra and geometry. We start with a classic example.

**Example 7**

Compute $\cos 40^\circ + \cos 80^\circ + \cdots + \cos 320^\circ$.

**Walkthrough** This is a good example if you're new to complex numbers.

1. What are the roots of $x^9 - 1$?
2. Divide $x^9 - 1$ by $x - (\cos 0^\circ + i \sin 0^\circ)$.
3. Use Vieta to finish. The answer is -1.

**Example 8**

Find the value of $\cos 72^\circ$.

**Walkthrough** Some students just try to memorize this value, but it's useful if you know how to derive it!

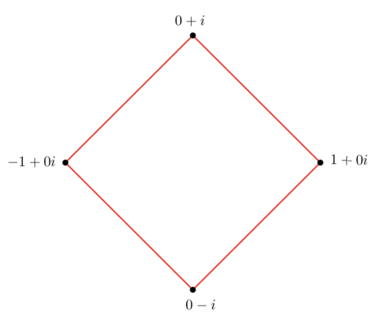
1. Let $z = e^{2\pi i/5}$. Find the equation (of degree 4) that $z$ satisfies.
2. Express $2\cos 72^\circ$ in terms of $z$.
3. Solve the equation and finish. The answer is $(-1+\sqrt{5})/4$.

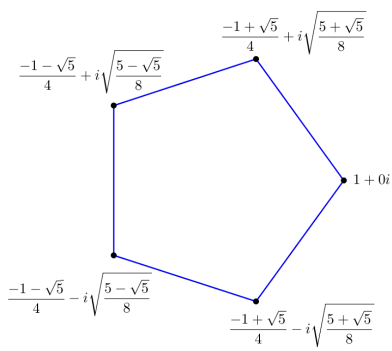
**Example 9 (2017 AMC 12A P17)**

There are 24 different complex numbers $z$ such that $z^{24} = 1$. For how many of these is $z^6$ a real number?

**Walkthrough** 17 17

1. What are the 24 different $z$?
2. Express $z^6$ with respect to $0 \le k \le 23$.
3. Finish. The answer is 12.



We now try to plot the roots of unity on the complex plane. These numbers obviously lie on the unit circle with radius 1 because all the roots have norm 1. The figure above shows a square and a regular pentagon formed by the 4th and 5th root of unity.

It is not hard to see that these points form a regular heptagon. To generalize:

**Theorem 4: Roots of Unity form a Regular Polygon**

The $n$th roots of unity form a regular $n$-gon on the complex plane.

**Example 10 (OTIS - Cyclotomic)**

Let $n \geq 3$ be a positive integer. Let $A_1A_2\dots A_n$ be a regular polygon inscribed in a circle of radius 1. Find the value of

$$A_nA_1 \times A_nA_2 \times A_nA_{n-1}.$$

**Walkthrough** Problem from Cyclotomic from OTIS. Let $\omega = e^{2\pi i/n}$.

1. Fix point $A_n$ at $(1,0)$. Why is this possible?
2. Let $A_1 = \omega$. Show that the length of $A_nA_1$ is $|1-\omega|$.
3. Consider the polynomial $(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$. How can you simplify this polynomial?
4. Plug in $x = 1$ and finish. The answer is $n$.

## 6 Vectors

Complex numbers can also be interpreted as vectors. For instance, $z = a + bi$ corresponds to the vector $\langle a, b\rangle$. But why?

Consider $P(a,b)$ and $Q(c,d)$. Then the vector $\overrightarrow{PQ}$ is $\langle c-a, d-b\rangle$. Now, let $p = a+bi$ and $q = c+di$. Then we get $q-p = (c-a)+(d-b)i$. See what's going on? We need to add $\overrightarrow{PQ}$ to go from $P$ to $Q$, and we need to add $q-p$ to move point $p$ to $q$!

To interpret complex numbers as vectors, suppose $P(a,b)$ and $p = a+bi$. Then, we can write $\overrightarrow{p} = \langle a, b\rangle$, which is a vector with initial point at the origin.

Therefore, some vector identities are also true when written by complex numbers! The most basic and well-known example is about the centroid of a triangle.

**Theorem 5: Centroid of a Triangle**

Let $a$, $b$, $c$ be the complex numbers corresponding to points of a triangle. Then, the centroid of the triangle corresponds to the vector $\frac{1}{3}(a+b+c)$.

**Example 11 (2012 AIME I P14)**

Complex numbers $a$, $b$ and $c$ are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to $a$, $b$, and $c$ in the complex plane are the vertices of a right triangle with hypotenuse $h$. Find $h^2$.

**Walkthrough** This is a very good example of alg and geo combined.

1. Deduce that the centroid of the triangle is at the origin.
2. Use Stewart's theorem to get the relationship between $|a|$, $|b|$, $|c|$, and the side lengths.
3. Finish. The answer is 375.

**Example 12 (2001 AIME II P14)**

There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

**Walkthrough** This problem can be solved without vectors, but it's more clear if you use vectors.

1. Prove that $z^{28}$ and $z^8$ should have the same imaginary part.
2. Prove that the vertices $O$, $z^8$, and $z^{28}$ form an equilateral triangle.
3. What values mod $2\pi$ can the argument of $z^8$ and $z^{28}$ have?
4. Finish. The answer is 840.

## 7 Rotations

Complex numbers are useful in geometry especially when rotations are involved. How do we rotate a point? We consider the complex numbers as vectors.

**Theorem 6: Rotating a Point**

Let $z$ be a complex number. The point obtained by rotating $z$ $\theta$ radians counterclockwise around the origin has complex number $z \cdot e^{i\theta}$.

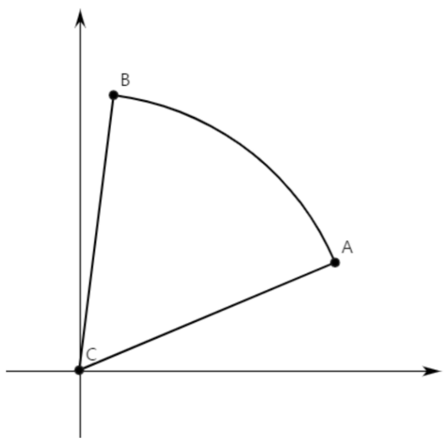
**Corollary : Rotating a Point, generalized**

Let $z$ be a complex number. The point obtained by rotating $z$ $\theta$ radians counterclockwise around the point $\omega$ has complex number $e^{i\theta}(z-\omega)+\omega)$.

Rotating a point around a point that is not the origin comes out very rarely on computational competitions, so the readers don't have to memorize the formula. To derive, shift points so that $\omega$ moves to the origin, rotate, and move $\omega$ back to where it was.

**Example 13 (Albert Shim)**

As in the figure above, an equilateral triangle $OAB$ is given on the first quadrant of the $xy$-plane. If point $O$ is the origin, the coordinate of vertex $A$ is $(8, p)$ and the equation of line $OB$ is $y = 8x$, what is the value of $p$?




**Walkthrough** Needs some bashing, sadly

1. Write $A$ as a complex number.
2. Find the complex number corresponding to $B$.
3. Finish. The answer is $\frac{64-8\sqrt{3}}{8\sqrt{3}+1}$.

**Example 14 (2014 AIME II P10)**

Let $z$ be a complex number with $|z| = 2014$. Let $P$ be the polygon in the complex plane whose vertices are $z$ and every $w$ such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Then the area enclosed by $P$ can be written in the form $n\sqrt{3}$, where $n$ is an integer. Find the remainder when $n$ is divided by 1000.

**Walkthrough** The rotation part comes at the very last.

1. Rearrange the equation, and get a quadratic equation in terms of $\omega/z$.
2. Solve the quadratic equation.
3. Fix $z$. What points can $\omega$ be?
4. Finish. The solution is 147.

# 8 Solutions to Examples

> **Example 2 (2005 AIME II P9)**
> For how many positive integers $n$ less than or equal to 1000 is
> $$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$
> true for all real $t$?

**Solution** Since $\cos t = \sin(\frac{\pi}{2} - t)$ and $\sin t = \cos(\frac{\pi}{2} - t)$, the equation can be changed to

$$\begin{aligned} \sin nt + i\cos nt &= (\sin t + i\cos t)^n \\ &= \left(\cos\left(\frac{\pi}{2} - t\right) + i\sin\left(\frac{\pi}{2} - t\right)\right)^n \\ &= \cos\left(\frac{n\pi}{2} - nt\right) + i\sin\left(\frac{n\pi}{2} - nt\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sin nt &= \cos\left(\frac{n\pi}{2} - nt\right) \quad \text{and} \\ \cos nt &= \sin\left(\frac{n\pi}{2} - nt\right). \end{aligned}$$

Since

$$\cos\left(\frac{n\pi}{2} - nt\right) = \sin nt = \cos\left(\frac{\pi}{2} - nt\right),$$

We have

$$\frac{n\pi}{2} - nt \equiv \frac{\pi}{2} - nt \pmod{2\pi}.$$

Therefore, $n \equiv 1 \pmod 4$, so there are 250 such $n$.

> **Example 3 (AMC 12B 2021 Spring P18)**
> Let $z$ be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

**Solution** Changing $|z|^2$ to $z\bar{z}$, we get

$$\begin{aligned} 12z\bar{z} &= 2(z+2)(\overline{z+2}) + (z^2+1)(\overline{z^2+1}) + 31 \\ &= 2(z+2)(\bar{z}+2) + (z^2+1)(\bar{z}^2+1) + 31. \end{aligned}$$

Expanding and rearranging, we get

$$(z\bar{z}-6)^2 + (z+\bar{z}+2)^2 = 0.$$

Since $z\bar{z}$ and $z+\bar{z}$ are both real, we have

$$z\bar{z}-6=0$$
$$z+\bar{z}+2=0.$$

Therefore, $z+\frac{6}{z} = z+\bar{z} = -2$.

**Example 4 (AMC 12B 2021 Fall P21)**

For real numbers $x$, let

$$P(x) = 1+\cos(x)+i\sin(x)-\cos(2x)-i\sin(2x)+\cos(3x)+i\sin(3x)$$

where $i=\sqrt{-1}$. For how many values of $x$ with $0 \le x < 2\pi$ does $P(x)=0$?

**Solution** Let $z=\cos x + i\sin x$. Then, $P(x) = 1+z-z^2+z^3$. We now need to find the number of roots of $P(x)$ with norm 1.

By the intermediate value theorem, since $P(-1)<0$ and $P(0)>0$, there is a real root $k$ strictly between $-1$ and $0$. Since this is the only real root, there are complex conjugate roots, $z$ and $\bar{z}$.

By Vieta, we have $kz\bar{z}=-1$. Since $-1<k<0$, $z\bar{z}=|z|^2>1$. Therefore, there are no roots with norm 1.

**Example 5**

Find $\cos 4\theta$ in terms of $\cos\theta$.

**Solution** Since $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$, $e^{i\theta}+e^{-i\theta} = 2\cos\theta$. Squaring both sides, we have

$$\begin{aligned} 4\cos^2\theta &= (e^{i\theta}+e^{-i\theta})^2 \\ &= e^{2i\theta}+e^{-2i\theta}+2, \end{aligned}$$

so $e^{2i\theta}+e^{-2i\theta} = 4\cos^2\theta - 2$. Again, squaring both sides gives

$$\begin{aligned} 16\cos^4\theta - 16\cos^2\theta + 4 &= (e^{2i\theta}+e^{-2i\theta})^2 \\ &= e^{4i\theta}+e^{-4i\theta}+2. \end{aligned}$$

Therefore, $\cos 4\theta = (e^{4i\theta}+e^{-4i\theta})/2 = 8\cos^4\theta - 8\cos^2\theta + 1$.

**Example 6 (AMC 12A 2021 Spring P22)**

Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is $abc$?

**Solution** Let $\omega = e^{2i\pi/7}$. We have

$$\begin{aligned} \cos\frac{2\pi}{7} &= \frac{\omega+\omega^{-1}}{2} \\ \cos\frac{4\pi}{7} &= \frac{\omega^2+\omega^{-2}}{2} \\ \cos\frac{6\pi}{7} &= \frac{\omega^3+\omega^{-3}}{2}, \end{aligned}$$

and $1+\omega+\omega^2+\cdots+\omega^6 = 0$. Now, by Vieta, we have

$$\begin{aligned} a &= -\frac{\omega+\omega^2+\omega^3+\omega^{-1}+\omega^{-2}+\omega^{-3}}{2} \\ &= -\frac{\omega+\omega^2+\cdots+\omega^6}{2} \\ &= \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} b &= \frac{\omega^3+\omega+\omega^{-1}+\omega^{-3}}{4} + \frac{\omega^5+\omega+\omega^{-1}+\omega^{-5}}{4} + \frac{\omega^4+\omega^2+\omega^{-2}+\omega^{-4}}{4} \\ &= \frac{2(\omega+\omega^2+\cdots+\omega^6)}{4} \\ &= -\frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} c &= -\frac{\omega^6+\omega^4+\omega^2+\omega^0+\omega^0+\omega^{-2}+\omega^{-4}+\omega^{-6}}{8} \\ &= -\frac{(\omega^6+\omega^5+\cdots+\omega^0)+\omega^0}{8} \\ &= -\frac{1}{8}. \end{aligned}$$

Therefore, $abc = 1/32$.

**Example 7**

Compute $\cos 40^\circ + \cos 80^\circ + \cdots + \cos 320^\circ$.

**Solution** Let $\omega = \cos 40^\circ + i\sin^\circ$. Then, $\omega$, $\omega^2$, $\ldots$, $\omega^9 = 1$ are roots to the equation $x^9 = 1$. So

$$x^9 - 1 = (x-\omega)(x-\omega^2)\cdots(x-\omega^8)(x-1),$$

and

$$(x-\omega)(x-\omega^2)\cdots(x-\omega^8) = \frac{x^9-1}{x-1} = x^8 + x^7 + \cdots + x + 1.$$

Therefore,

$$\begin{aligned}\cos 40^\circ + \cos 80^\circ + \cdots + \cos 320^\circ &= \Re(\omega + \omega^2 + \cdots + \omega^8)\\ &= \Re(-1) = -1\end{aligned}$$

by Vieta.

**Example 8**

Find the value of $\cos 72^\circ$.

**Solution** Let $z = e^{2\pi i/5}$. Then, $z^5 - 1 = 0$. Since $z \neq 1$, we can divide the equation by $z - 1$, which gives

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Let $t = 2\cos 72^\circ = z + \frac{1}{z}$. Then, dividing the whole equation by $z^2$,

$$\begin{aligned}z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} &= \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right) - 1\\ &= t^2 + t - 1 = 0.\end{aligned}$$

Solving, we get $t = 2\cos 72^\circ = (-1+\sqrt{5})/2$. Therefore, $\cos 72^\circ = (-1+\sqrt{5})/4$.

**Example 9 (2017 AMC 12A P17)**

There are 24 different complex numbers $z$ such that $z^{24} = 1$. For how many of these is $z^6$ a real number?

**Solution** The 24 different complex numbers are $e^{k\pi i/12}$ where $k = 0$, $1$, $\ldots$, $23$. To make $z^6 = e^{6k\pi i/12}$ real, $6k/12$ should be an integer, so $k = 0$, $2$, $\ldots$, $22$. Therefore, there are 12 such complex numbers.

**Example 10 (OTIS - Cyclotomic)**

Let $n \geq 3$ be a positive integer. Let $A_1A_2 \dots A_n$ be a regular polygon inscribed in a circle of radius 1. Find the value of

$$A_nA_1 \times A_nA_2 \times \cdot A_nA_{n-1}.$$

**Solution** Let $\omega = e^{2\pi i/n}$, and plot $A_i$ at $\omega^i$ on the complex plane. Then, the length of $A_nA_i$ is $|1 - \omega^i|$. So the desired value is

$$|1-\omega| \cdot |1-\omega^2| \cdot \dots \cdot |1-\omega^{n-1}|.$$

Since the polynomial with roots $\omega$, $\omega^2$, $\dots$, $\omega^{n-1}$ is

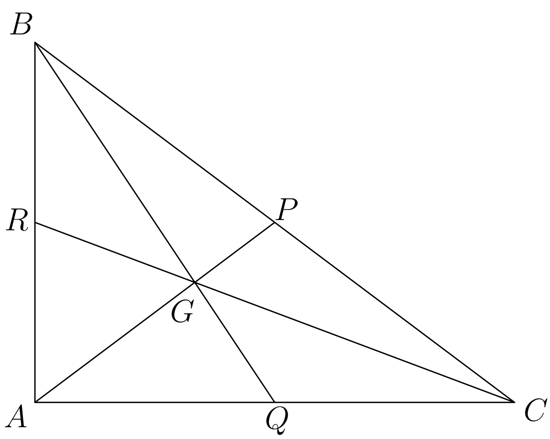
$$\begin{aligned} P(x) &= (x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}) \\ &= x^{n-1} + x^{n-2} + \dots + x + 1, \end{aligned}$$

the desired value is $|P(1)|$, which is $n$.

**Example 11 (2012 AIME I P14)**

Complex numbers $a$, $b$ and $c$ are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to $a$, $b$, and $c$ in the complex plane are the vertices of a right triangle with hypotenuse $h$. Find $h^2$.

**Solution** By Vieta, we have $a + b + c = 0$. Since $\frac{1}{3}(a+b+c) = 0$, the centroid of the triangle is the origin.



We now have $AG^2 + BG^2 + CG^2 = 250$. By Stewart, we get

$$\begin{aligned} BC^2 &= 4BP^2 \\ &= 2(BG^2 + CG^2) - 4GP^2 \\ &= 2(BG^2 + CG^2) - AG^2. \end{aligned}$$

Similarly,

$$\begin{aligned} CA^2 &= 2(CG^2 + AG^2) - BG^2 \\ AB^2 &= 2(AG^2 + BG^2) - CG^2. \end{aligned}$$

Adding the three equations we get

$$\begin{aligned} AB^2 + BC^2 + CA^2 &= 2h^2 \\ &= 3(AG^2 + BG^2 + CG^2) \\ &= 750, \end{aligned}$$

so $h^2 = 375$.

**Example 12 (2001 AIME II P14)**

There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

**Solution** Consider a vector from $z^8$ to $z^{28}$. This vector is $\langle 1, 0\rangle$. Therefore, $z^{28}$ and $z^8$ should have the same imaginary part. Now, since $z^{28}$ and $z^8$ are on the unit circle, $|z^{28}| = |z^8| = 1$. So the vertices $O$, $z^8$, and $z^{28}$ form an equilateral triangle. Let $z = e^{i\theta}$. Then, we should have

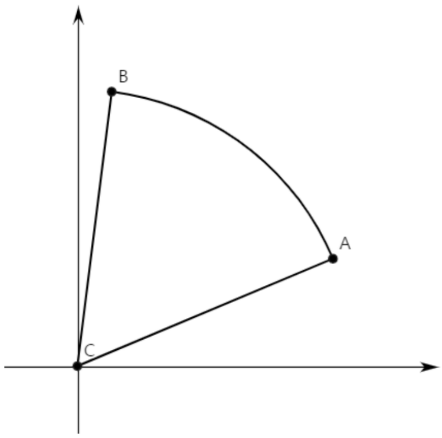
$$\begin{aligned} 28\theta &\equiv \pm 60^\circ \pmod{2\pi}, \text{ and} \\ 8\theta &\equiv \pm 120^\circ \pmod{2\pi}. \end{aligned}$$

Therefore, $\theta$ can be $15^\circ$, $75^\circ$, $105^\circ$, $165^\circ$, $195^\circ$, $255^\circ$, $285^\circ$, and $345^\circ$. The desired value is

$$75 + 165 + 255 + 345 = 840.$$

**Example 13 (Albert Shim)**

As in the figure above, an equilateral triangle $OAB$ is given on the first quadrant of the $xy$-plane. If point $O$ is the origin, the coordinate of vertex $A$ is $(8, p)$ and the equation of line $OB$ is $y = 8x$, what is the value of $p$?




**Solution** Let $a = 8 + pi$ be the complex number corresponding to $A$. Then, since $OA = OB$ and $\angle AOB = 60°$, we have

$$\begin{aligned} b &= a \cdot e^{i \cdot 60°} \\ &= (8+pi)\left(\frac{1+\sqrt{3}i}{2}\right) \\ &= \frac{(8-\sqrt{3}p)+(p+8\sqrt{3})i}{2}, \end{aligned}$$

where $b$ is the complex number corresponding to $B$. Since $B$ is on the line $y = 8x$,

$$\frac{p+8\sqrt{3}}{2} = 8 \cdot \frac{8-\sqrt{3}p}{2}.$$

Therefore, we have $p = \frac{64-8\sqrt{3}}{8\sqrt{3}+1}$.

**Example 14 (2014 AIME II P10)**

Let $z$ be a complex number with $|z| = 2014$. Let $P$ be the polygon in the complex plane whose vertices are $z$ and every $w$ such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Then the area enclosed by $P$ can be written in the form $n\sqrt{3}$, where $n$ is an integer. Find the remainder when $n$ is divided by 1000.

**Solution** Rearranging the equation gives

$$z^2 + z\omega + \omega^2 = 0.$$

Dividing by $^2$, we get a quadratic equation:

$$\left(\frac{\omega}{z}\right)^2 + \left(\frac{\omega}{z}\right) + 1 = 0.$$

Solving, we get $\omega/z = e^{2\pi i/3}$, $e^{4\pi i/3}$. Fixing $z$ at $(2014, 0)$, we get $\omega = 2014e^{2\pi i/3}$, $2014e^{4\pi i/3}$. Therefore, $P$ is an equilateral triangle with side length $2014\sqrt{3}$. The area of $P$ is $3 \cdot 1007^2\sqrt{3}$, so $3 \cdot 1007^2 \equiv 3 \cdot 7^2 = 147 \pmod{1000}$.

# 9 Problems

## Basics

Problems here can be solved only using the basic properties of complex numbers.

**Problem 1 (2018 AIME II P5)**

Suppose that $x$, $y$, and $z$ are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers $a$ and $b$ such that $x + y + z = a + bi$. Find $a^2 + b^2$.

**Problem 2 (2010 AIME II P7)**

Let $P(z) = z^3 + az^2 + bz + c$, where $a$, $b$, and $c$ are real. There exists a complex number $w$ such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

**Problem 3 (2012 AIME II P8)**

The complex numbers $z$ and $w$ satisfy the system

$$z + \frac{20i}{w} = 5 + i$$

$$w + \frac{12i}{z} = -4 + 10i.$$

Find the smallest possible value of $|zw|^2$.

## De Moivre's Theorem

**Problem 4 (2022 AIME I P4)**

Let $w = \dfrac{\sqrt{3} + i}{2}$ and $z = \dfrac{-1 + i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs $(r, s)$ of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

**Problem 5 (2016 AIME I P6)**

The complex numbers $z$ and $w$ satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of $z$ is $\sin\left(\frac{m\pi}{n}\right)$ for relatively prime positive integers $m$ and $n$ with $m < n$. Find $n$.

**Problem 6 (2019 AIME II P10)**

There is a unique angle $\theta$ between $0°$ and $90°$ such that for nonnegative integers $n$, the value of $\tan(2^n\theta)$ is positive when $n$ is a multiple of 3, and negative otherwise. The degree measure of $\theta$ is $\frac{p}{q}$, where $p$ and $q$ are relatively prime integers. Find $p + q$.

## Trigonometry

**Problem 7 (2000 AIME II P9)**

Given that $z$ is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

**Problem 8 (PUMaC 2010 Division A Algebra P7)**

The expression $\sin 2° \sin 4° \sin 6° \cdots \sin 90°$ is equal to $p\sqrt{5}/2^{50}$, where $p$ is an integer. Find $p$.

## Roots of Unity

**Problem 9 (1990 AIME P10)**

The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in $C$?

**Problem 10 (2019 AIME II P8)**

The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f(\frac{1+\sqrt{3}i}{2}) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

**Problem 11 (2004 AIME I P13)**

The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k\big(\cos(2\pi a_k) + i\sin(2\pi a_k)\big), k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

## Rotations

**Problem 12 (2014 AIME I P7)**

Let $w$ and $z$ be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. The maximum possible value of $\tan^2\theta$ can be written as $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to $w$ makes with the positive real axis in the complex plane.

**Problem 13 (2015 AIME II P13)**

Define the sequence $a_1, a_2, a_3, \ldots$ by $a_n = \sum_{k=1}^{n} \sin(k)$, where $k$ represents radian measure. Find the index of the 100th term for which $a_n < 0$.

## Vectors

**Problem 14 (2018 AIME I P6)**

Let $N$ be the number of complex numbers $z$ with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when $N$ is divided by 1000.

## Mixed Practice

**Problem 15 (2023 AIME II P8)**

Let $\omega = \cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7}$, where $i = \sqrt{-1}$. Find

$$\prod_{k=0}^{6} (\omega^{3k} + \omega^k + 1).$$

**Problem 16 (2020 AIME I P8)**

A bug walks all day and sleeps all night. On the first day, it starts at point $O$, faces east, and walks a distance of 5 units due east. Each night the bug rotates $60°$ counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point $P$. Then $OP^2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**Problem 17 (Math Prize for Girls 2016 P12)**

Let $b_1$, $b_2$, $b_3$, $c_1$, $c_2$, and $c_3$ be real numbers such that for every real number $x$, we have

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)(x^2 + b_3x + c_3).$$

Compute $b_1c_1 + b_2c_2 + b_3c_3$.

**Problem 18 (1994 AIME P13)**

The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

**Problem 19 (2013 AIME I P14)**

For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \dots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \dots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**Problem 20 (Math Prize for Girls 2015 P15)**

Let $z_1$, $z_2$, $z_3$, and $z_4$ be the four distinct complex solutions of the equation

$$z^4 - 6z^2 + 8z + 1 = -4(z^3 - z + 2)i.$$

Find the sum of the six pairwise distances between $z_1$, $z_2$, $z_3$, and $z_4$.

**Problem 21 (Math Prize for Girls 2012 P13)**

For how many integers $n$ with $1 \le n \le 2012$ is the product

$$\prod_{k=0}^{n-1} \left( \left(1 + e^{2\pi ik/n}\right)^n + 1 \right)$$

equal to zero?

## Answers

- Problem 1 (2018 AIME II P5): 074
- Problem 2 (2010 AIME II P7): 136
- Problem 3 (2012 AIME II P8): 040
- Problem 4 (2022 AIME I P4): 834
- Problem 5 (2016 AIME I P6): 071
- Problem 6 (2019 AIME II P10): 547
- Problem 7 (2000 AIME II P9): 000
- Problem 8 (PUMaC 2010 Division A Algebra P7): 192
- Problem 9 (1990 AIME P10): 144
- Problem 10 (2019 AIME II P8): 053
- Problem 11 (2004 AIME I P13): 482
- Problem 12 (2014 AIME I P7): 100
- Problem 13 (2015 AIME II P13): 628
- Problem 14 (2018 AIME I P6): 440
- Problem 15 (2023 AIME II P8): 024
- Problem 16 (2020 AIME I P8): 103
- Problem 17 (Math Prize for Girls 2016 P12): -1
- Problem 18 (1994 AIME P13): 850
- Probelm 19 (2013 AIME I P14): 036
- Problem 20 (Math Prize for Girls 2015 P15): $6 + 6\sqrt{3}$
- Problem 21 (Math Prize for Girls 2012 P13): 670