Complex-variable Functions

PROMYS Minicourse

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1 Basics

This is a quick review of complex numbers, and it is assumed that the reader know the basics of complex numbers: arithmetic, polar form, etc.

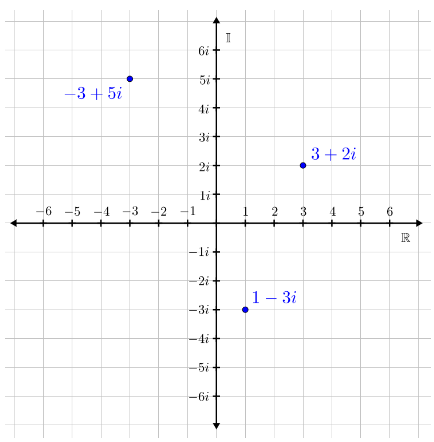
1.1 The Complex Plane

Definition 1.1: Complex Number

A **complex number** z is a number of the form

$$z = a + bi$$

where a and b are real numbers, and $i = \sqrt{-1}$.



We wish to interpret these numbers geometrically. Recall that we plot real numbers in a line. We can do a similar thing for complex numbers. Let x-axis be the real axis, and y-axis be the imaginary axis. Then, for any complex number $a+bi$, there is exactly one point corresponding, namely (a, b). The figure above shows $-3+5i$ at $(-3, 5)$, $3+2i$ at $(3, 2)$, and $1-3i$ at $(1, -3)$.

Recall that an absolute value of a real number x is the distance from 0 to x on the real line. Similarly, an absolute value of z can be defined as the distance from 0 to z on the complex plane, which is $\sqrt{a^2+b^2}$.

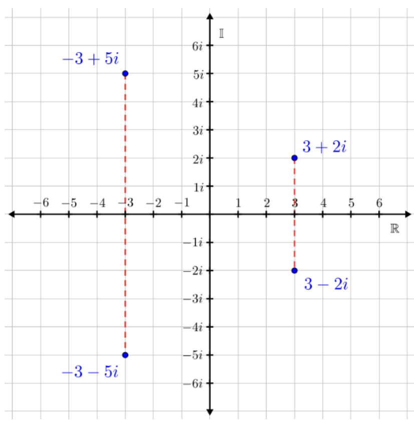
Definition 1.2: Modulus

A **modulus** of a complex number is

$$|z| = \sqrt{a^2+b^2}.$$

The term modulus, magnitude, and norm are all equal, and may be used interchangeably.

The complex conjugate can also be thought of as the reflection of a complex number about the real axis in the complex plane. The figure shows the points corresponding to $-3+5i$, $3+2i$, and their complex conjugates.



1.2 Complex-Variable Functions

Definition 1.3: Complex-Variable Functions

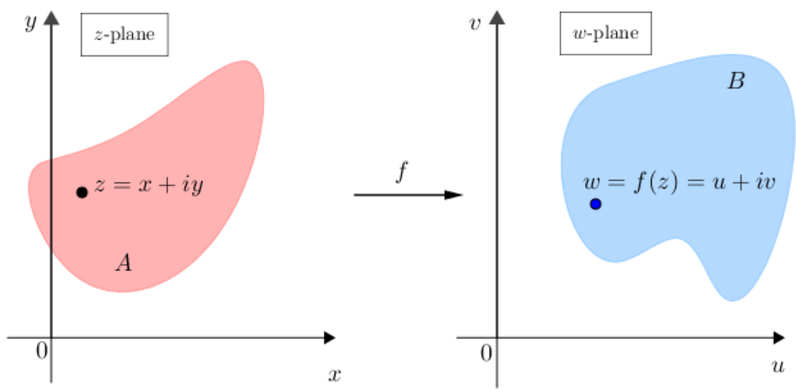
A function $f : \mathbb{C} \to \mathbb{C}$ with variable $z \in \mathbb{C}$ is called a complex-variable function.

Example 1

If $f(z) = z^2$, then $f(1+i) = (1+i)^2 = 2i$, and $f(1) = 1^2 = 1$.

2 Visualizing the Function

How do we draw these functions? For real functions $f : \mathbb{R} \to \mathbb{R}$, we used one axis (the x-axis) for the domain, and another axis (the y-axis) for the range. We could not do such thing in complex functions! Since every complex number is written as $a + bi$ where a and b are real numbers, you need two real numbers to visualize one complex number. Thus you need two axes for the complex numbers in the domain, and other two for the ones in the range, which sums to 4. So we could not visualize complex functions as the way we did in real functions since we can only visualize up to 3-dimensions. We thus treat the domain and the range separately.



If $f : A \to B$ where A and B are open subsets of $\mathbb{C}$, by f, $z = x + iy$ is mapped to $w = u + iv$. We also write the function as $w = f(z)$, as an analogue of $y = f(x)$.

Because the domain and range is drawn separately, an alternative is to understand complex functions by tracking how they transform curves and regions in the input plane. In particular, we focus on how lines and rays (horizontal/vertical lines, radial lines, circles) in the domain are mapped under the function. This gives an idea of the behavior of the function, without requiring a full 4D picture.

3 Linear Mappings

From now on, we use either *mappings* or *transformations* for complex functions. The three most basic complex mappings are:

- Translation: $\omega = f(z) = z + b$ where $b \in \mathbb{C}$
- Rotation: $\omega = az$, $a = e^{i\alpha}$

- Dilation: $\omega = az$, $a > 0$, $a \neq 1$

If we have $b = c + id$, then $\omega = z + b$ translates the point c units to the right, d units to the up (note that c and d may be negative, if $c < 0$, then z is moved $-c$ units left, and if $d < 0$, then z is moved $-d$ units down).

The rotation $\omega = e^{i\alpha}z$ rotates z the amount of α with respect to the origin. Since one rotation has angle 2π, the rotation $\omega = e^{i(2\pi)}z$ is the identity rotation, which makes sense since $e^{2\pi i} = 1$.

Dilation changes the distance between z and the origin, and preserves the angle.

For the first three, we could simply write $f(z) = az + b$ where $a, b \in \mathbb{C}$, which is a combination of scaling and rotation, followed by a translation. So geometrically,

- $|a|$ is the scaling factor,
- $\arg a$ is the rotation angle (counterclockwise),
- b is the translation vector.

Here, if $a = x + iy$, then $\arg a$ is defined as

$$\arg a = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x < 0 \\ \frac{\pi}{2} & x = 0,\ y > 0 \\ -\frac{\pi}{2} & x = 0,\ y < 0. \end{cases}$$

Such mappings are called linear mappings. Note that these mappings not linear in the real-variable sense, and is different from linear transformations.

4 Inversion

Besides the first three basic mappings, there is one more basic mapping: the *inversion*. The inversion is the mapping $\omega = 1/z = e^{-i\theta}/r$. The inversion can be interpreted as a reflection with respect to the unit circle and the real axis, since the new norm became the reciprocal to the original, and the angle became negative.

By writing

$$\omega = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2},$$

there is a bijection between $x + iy$ and its inversion, $(x, y) \leftrightarrow (u, v)$ where $u = \dfrac{x}{x^2 + y^2}$ and $v = -\dfrac{y}{x^2 + y^2}$.

Suppose $a(x^2 + y^2) + bx + cy + d = 0$ for $z = (x, y)$. Then, this is a straight line if

$a = 0$ and circle if $a \neq 0$. Dividing both sides by $x^2 + y^2$,

$$a + \frac{bx}{x^2 + y^2} + \frac{cy}{x^2 + y^2} + \frac{d}{x^2 + y^2} = 0$$

Mapping this by $\omega = \frac{1}{z}$ is just rewriting the relation with respect to (u, v).

$$a + bu - cv + d(u^2 + v^2) = 0$$

This is a straight line it $d = 0$, and a circle if $d \neq 0$. Then, there are four cases.

- If a, $d \neq 0$, then the inversion maps a circle not passing the origin to another (not necessarily distinct) circle not passing the origin.
- If $a \neq 0$ and $d = 0$, then the inversion maps a circle passing the origin to a straight line not passing the origin.
- If $a = 0$ and $d \neq 0$, then the inversion maps a straight line not passing the origin to a circle passing the origin.
- If $a = d = 0$, then the inversion maps a straight line passing the origin to another (not necessarily distinct) straight line passing the origin.

This is summarized to the following theorem.

Theorem 4.1

The inversion $\omega = 1/z$ maps
{circles and straight lines} onto {circles and straight lines}.

5 Möbius Transformations

Consider the rational function $f(z) = \dfrac{az + b}{cz + d}$ where a, b, c, and $d \in \mathbb{C}$. This is almost a bijective map from $\mathbb{C}$ to $\mathbb{C}$, with two problems:

- How should we define $f(-d/c)$?
- For what z is $f(z) = c/a$?

We see that both problems involve infinity. If we assume that there is an ideal point ∞ at infinity, we could define the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Definition 5.1: Möbius Transformation

A **Möbius transformation** is a rational function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$.

Then we have

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & z \neq -\dfrac{d}{c}, \infty \\ \infty & z = -\dfrac{d}{c} \\ \dfrac{a}{c} & z = \infty \end{cases},$$

so Möbius transformations are bijections from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$.

Exercise 1

Why do we consider only $ad - bc \neq 0$?

It could be deduced that all Möbius transformations are compositions of basic mappings. If $T(z) = \frac{az+b}{cz+d}$, then $T(z) = f_3 \circ f_2 \circ f_1$ where

$$f_1 = z + \frac{d}{c},\ f_2 = \frac{1}{z},\ f_3 = \frac{a}{c} - \left(\frac{ad-bc}{c^2}\right) z.$$

This gives the following theorem.

Theorem 5.1

The bilinear transformation maps
{circles and straight lines} onto {circles and straight lines}.

Example 2

Show that $\Re\left(\dfrac{z}{1-z}\right) > -\dfrac{1}{2}$ if $|z| < 1$.

Solution We have $\dfrac{z}{1-z} = f_3 \circ f_2 \circ f_1$ where $f_1 = 1 - z$, $f_2 = \dfrac{1}{z}$, and $f_3 = z - 1$. The region of f_1 maps $|z| < 1$ to $|1 - z| < 1$, which is $x^2 + y^2 - 2x < 0$. With f_2, we have

$$\begin{aligned} \frac{1}{x^2+y^2}(x^2 + y^2 - 2x) &< 0 \\ 1 - \frac{2x}{x^2+y^2} &< 0 \\ 1 - 2u &< 0 \\ u &> \frac{1}{2}. \end{aligned}$$

Finally, f_3 gives $u > -\dfrac{1}{2}$, so $\Re\left(\dfrac{z}{1-z}\right) > -\dfrac{1}{2}$.

We look at the graph of Möbius transformations. As we said earlier, we are particularly interested in how horizontal and vertical lines are mapped under the function.

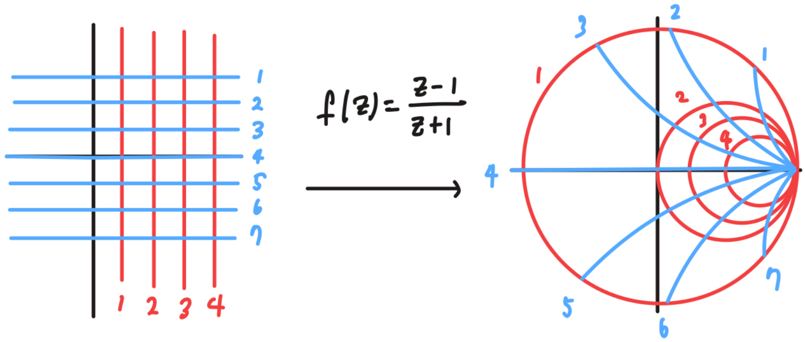


Figure 1: Mapping of $f(z) = \frac{z-1}{z+1}$

Consider the function $f(z) = \dfrac{z-1}{z+1}$. Under this function, the four red lines and the seven blue lines (line 4 is the x-axis) is mapped to the diagram on the right.

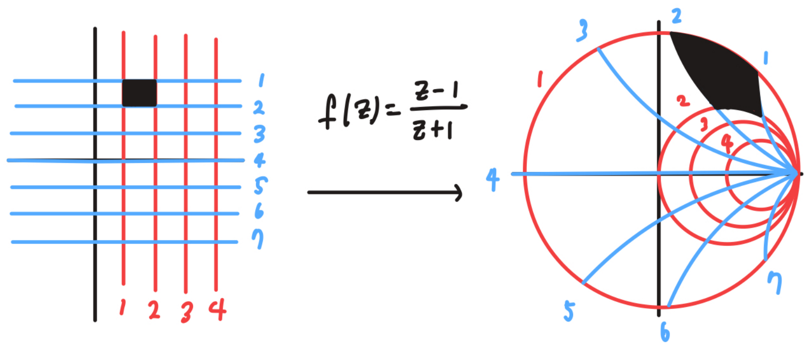


Figure 2: Mapping of $f(z) = \frac{z-1}{z+1}$

The region inside is preserved, so the region between red 1 and 2 and blue 1 and 2 on the domain is also the region between red 1, 2 and blue 1, 2 on the range (the black region on the domain is mapped to the black region on the range). If we want the map of a smaller region, we can divide lines more precisely.

6 Elementary Functions

6.1 Exponential Functions

Theorem 6.1: Euler's Equation

If x is a real number, then $e^{ix} = \cos x + i \sin x$.

Thus, for $z = x + iy$, define

$$e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y).$$

Then, we have the following exponent identities, as usual.

1. $e^{z_1}e^{z_2} = e^{z_1+z_2} = e^{x_1+x_2}\left(\cos(y_1 + y_2) + i \sin(y_1 + y_2)\right)$
2. $e^{z_1}/e^{z_2} = e^{z_1-z_2}$
3. $(e^z)^n = e^{nz}$ where $n \in \mathbb{Z}$
4. $e^z \neq 0$ for $\forall z \in \mathbb{C}$ since $|e^z| = e^x > 0$.

Remark.
e^z is not one-to-one, because $e^{z+2\pi ki} = e^z$. e^z is one-to-one mod $2\pi i$.

Theorem 6.2

The exponential function $f(z) = e^z$ is $2\pi i$-periodic, and is unique up to mod $2\pi i$.

Example 3
If $e^z = a + ib$, then what is z?

Solution We let $z = x + iy$, and $e^{x+iy} = a + ib$. Then,

$$a = e^x \cos y \text{ and } b = e^x \sin y.$$

Solving for x and y, we get

$$x = \frac{1}{2}\ln(a^2 + b^2) \text{ and}$$

$$\tan y = \frac{b}{a}, \text{ so}$$

$$y = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi & a > 0 \\ \tan^{-1}\left(\frac{b}{a}\right) + (2k+1)\pi & a < 0 \end{cases}.$$

Exercise 2

Find z where $e^z = 5 - 5i$.

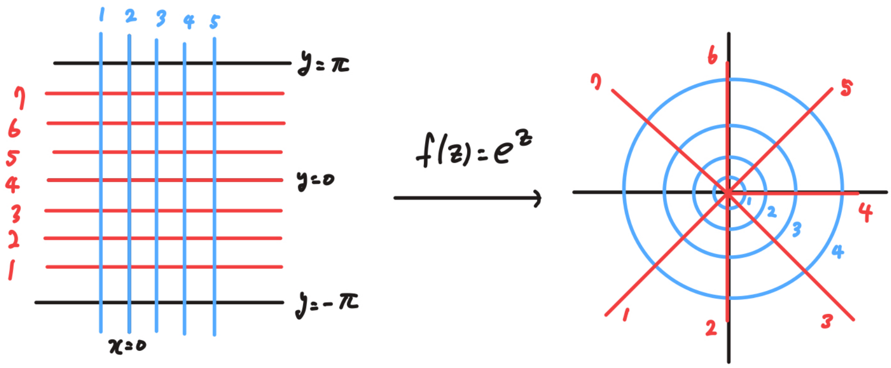


Figure 3: Mapping of $f(z) = e^z$

The mapping of the exponential function is drawn as follows. Since the exponential function is periodic mod $2\pi i$, we only consider $-\pi \le y \le \pi$. The horizontal lines ($y = k$) are mapped to rays starting from the origin (this is because as $x \to -\infty$, $e^x \to 0$), and the vertical lines ($x = k$) are mapped to circles (this is because the modulus of e^z is e^x, which is fixed).

We now look on trigonometric functions. Since

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \text{ and } \cos y = \frac{e^{iy} + e^{-iy}}{2},$$

We extend this relation to $\mathbb{C}$ and get

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos y = \frac{e^{iz} + e^{-iz}}{2}.$$

Some properties still hold in $\mathbb{C}$.

- $\sin z$ is an odd function, i.e. $\sin(-z) = -\sin z$.
- $\cos z$ is an even function, i.e. $\cos(-z) = \cos z$.
- $\sin z = 0 \Leftrightarrow z = k\pi \ (k \in \mathbb{Z})$
- $\cos z = 0 \Leftrightarrow z = (k + \frac{1}{2})\pi \ (k \in \mathbb{Z})$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\cos^2 z + \sin^2 z = 1$

We can also define other trigonometric functions:

$$\tan z = \frac{\sin z}{\cos z}$$
$$\sec z = \frac{1}{\cos z}$$
$$\csc z = \frac{1}{\sin z}$$
$$\cot z = \frac{1}{\tan z}$$
$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}.$$

Note that $\cosh z$ and $\sinh z$ are also $2\pi i$-periodic. Most of the properties are true also for complex variables, but $\sin z$ and $\cos z$ are not bounded anymore.

Exercise 3

Solve $\cos z = 2$ over the complex numbers.

6.2 Logarithmic Functions

Notice that for given $z \in \mathbb{C}$, $e^\omega = z$ does not have a unique solution, but it is unique up to mod $2\pi i$. Therefore, if ω_0 satisfies $e^{\omega_0} = z$, then

$$\{\omega \mid \omega_0 - \omega = 2k\pi i,\ k \in \mathbb{Z}\}$$

is the set of all ω satisfying $e^\omega = z$.

Definition 6.1: $\log z$

The function $\log z$ is a multi-valued function (or a set-valued function).

$$\log z = \{\omega \mid \omega = \ln|z| + i(\arg z + 2k\pi), k \in \mathbb{Z}\}$$

To get the formula of $\log z$, we set $z = re^{i\theta}$. Since $e^\omega = z$, $e^u e^{iv} = re^{i\theta}$, we have

$$e^u = r \quad \text{and} \quad e^{iv} = e^{i\theta}.$$

Therefore, $u = \ln r$ and $v = \theta + 2k\pi$ for $k \in \mathbb{Z}$.

Remark.

$\log 0$ is not defined since the range of e^z is $\mathbb{C} \setminus \{0\}$.

Remark.

For $x \in \mathbb{R}$, $e^x \to \infty$ as $x \to \infty$. However, this is not true for complex numbers. Note that we can obtain arbitrarily large modulus by keep adding $2\pi i$.

But we want $\log z$ to be one specific value, instead of a set of values. Thus we fix k as 0 in the definition of $\log z$, and we call this the principal logarithm.

Definition 6.2: Principlal Logarithm

The **principal logarithm** function is defined by

$$\operatorname{Log} z = \ln |z| + i \arg z$$

where $\arg z \in (-\pi, \pi]$.

6.3 Complex Exponents

Recall that $z = re^{i\theta} = re^{i(\theta+2k\pi)}$. Define z_k as complex numbers satisfying $z_k^n = z$. Then,

$$z_k = r^{1/n} e^{i(\theta+2k\pi)/n}$$

for $k = 0, \dots, n-1$, and they are all distinct. It is ambiguous to single out one of them as $z^{1/n}$.

Definition 6.3: $z^{1/n}$

The function $\omega = z^{1/n} = e^{(1/n)(\ln|z|+i\arg z)}$ is a multi-valued function, which has n distinct values.

Now, suppose $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. We consier z^α. We have

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{\alpha(\ln r + i(\theta+2k\pi))} \\ &= r^\alpha e^{i\alpha\theta} e^{i\cdot 2k\pi\alpha}. \end{aligned}$$

If we let $\alpha = a + ib$, then

$$\begin{aligned} z^\alpha &= e^{(a+ib)\log z} \\ &= e^{a\ln r - b(\theta+2k\pi)} \cdot e^{i(b\ln r + a\theta + 2k\pi a)}. \end{aligned}$$

The value varies for each k, and hence there are infinite numbers that can be z^α. Therefore, we should think the output of exponential functions as not a number, but a set of numbers.

Example 4

We have the following complex exponents:

$$\begin{aligned} 5^{1/2} &= e^{(1/2)\log 5} \\ &= e^{(1/2)(\ln 5 + 2k\pi i)} \\ &= \{\pm\sqrt{5}\} \\ i^i &= e^{i \log i} \\ &= e^{i(\ln 1 + i(\pi/2 + 2k\pi))} \\ &= \left\{e^{-\pi/2 + 2k\pi},\ k \in \mathbb{Z}\right\} \end{aligned}$$

Exercise 4

Find z such that $z^{1-i} = 4$.

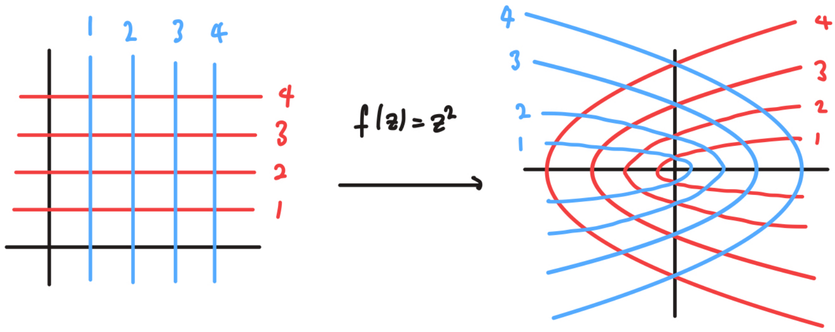


Figure 4: Mapping of $f(z) = z^2$

The function $f(z) = z^2$ squares horizontal and vertical rays to parabolas, as shown in the figure above.