Coordinate Geometry

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This handout covers coordinate bashing in geometry for computational competitions. Even though most of the computational competitions require fast problem-solving, but if there's a lot of time, then it's worth it to try bashing. So even if the problem didn't give you any information about Cartesian coordinates, it can be solved there.

Keep in mind: If a geometry problem is possible with coordinate bashing, then it is possible to solve (assuming there is enough time).

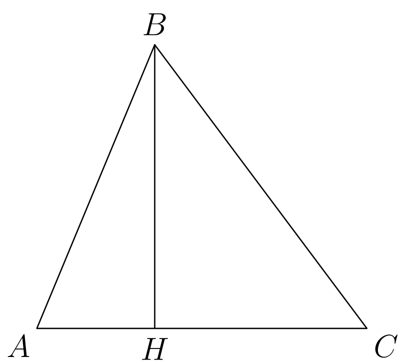
This handout is written assuming that the problem hasn't given any coordinate information and that readers already know most of the basic geometry.

1 How to Begin

Most geometry problems start with a triangle. If it doesn't, then it is mostly not suitable for coordinate bashing.

Note that we want to minimize our calculations. We now need to set the origin. Why can we do this?

Suppose that in a $13-14-15$ triangle, one is asked to find the altitude of the segment 14. (Note that this value is 12 since the area is 84 by Heron.)



Let $a = 15$, $b = 14$, and $c = 13$. We want to find the length of BH. To do this, we can set $A(a, b)$. Then C will have $(a + 14, b)$. We let $B(x, y)$, and can get a system of equations

$$(x - a)^2 + (y - b)^2 = 169$$
$$(x - a - 14)^2 + (y - b)^2 = 225.$$

This will give x and y (in terms of a and b), and we can find the coordinate of B.

This method will give the altitude, but the computation is too messy. Notice that the choice of a and b is independent to the length of BH. Therefore, we set $a = 0$ and $b = 0$, for easier calculations.

If $A(0, 0)$, then C will be $(14, 0)$. Let $B(x, y)$ again, then we get

$$x^2 + y^2 = 169$$
$$(x - 14)^2 + y^2 = 225.$$

Clean, right? Solving this system will give $x = 5$ and $y = 12$. Therefore, $B(5, 12)$, and H will be $(5, 0)$, so $BH = 12$.

With the previous example, the reader should have noticed that letting the most coordinates to zero will reduce the computation. At most times, three out of the six coordinates (the x and y coordinates of three points) can be zero. There are two cases:

Case 1: Letting one coordinate the origin, and one coordinate have a positive x-coordinate with y-coordinate zero.

This is the case what we did in finding BH. We let A the origin, and $B(14, 0)$ so that AB overlap with the x-coordinate.

Case 2: Let H be the origin and AC overlap the x-axis, where H is the feet of altitude from B to AC.

For example, if a $13 - 14 - 15$ triangle is given and the problem is asking something there, we can let $H(0, 0)$. This will give $A(-5, 0)$, $B(0, 12)$, and $C(9, 0)$.

To summarize, there are two points to consider when solving a problem on a Cartesian plane.

- Choosing the point to set as the origin
- Choosing the segment to overlap with the coordinate axis.

2 Theorems Used

Theorem : Slope and Tangents

Let l be the line on the Cartesian plane. If the angle between l and the x-axis is θ, then the slope of l is $\tan\theta$.

Theorem : Distance between Two Points

The distance between two points (x_1, y_1) and (x_2, y_2) are

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Theorem : Distance between a Point and a Line

The distance between a point (x_0, y_0) and the line $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Theorem : Distance between Two Parallel Lines

The distance between two lines $ax + by + c = 0$ and $ax + by + d = 0$ is

$$\frac{|c - d|}{\sqrt{a^2 + b^2}}.$$

Theorem : Equation of a Circle

A circle has an equation of two forms.

$$x^2 + y^2 + ax + by + c = 0$$

$$(x - a)^2 + (y - b)^2 = r^2.$$

The first one is useful if three points on the circle are given, and the second one is useful if some information about the center of the radius is given.

Theorem : Point on a Segment

Let AB be a segment with $A(x_1, y_1)$ and $B(x_2, y_2)$. If P is a point on segment AB such that $\frac{PA}{PB} = \frac{m}{n}$, then the coordinate of P is

$$P\left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right).$$

Theorem : Shoelace Theorem

Let $A_1A_2\cdots A_n$ be a convex n-gon. If $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $\ldots$, $A_n(x_n, y_n)$, then the area of this n-gon is

$$\frac{1}{2}\big((x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\big).$$

Here, letting $A_1(0, 0)$ will make the computation easier.

Corollary : Area of a Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Then the area of $\triangle ABC$ is the determinant of the matrix

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Theorem : Point Reflections

Let $P(a, b)$. Then the points obtained by the following lines have the following coordinates:

- x-axis: $(a, -b)$
- y-axis: $(-a, b)$
- Origin: $(-a, -b)$
- $y = x$: (b, a)
- $y = -x$: $(-b, -a)$

3 Useful Methods and Results to Know

Example 1

Find the point obtained by reflecting the point $(1, 2)$ with respect to the line $x + 2y - 3 = 0$.

Walkthrough

1. Let the reflected point be $P'(x, y)$.
2. Get an equation with the fact that the midpoint of PP' is on $x + 2y - 3 = 0$.
3. Get an another equation with the fact that the line PP' is perpendicular to $x + 2y - 3 = 0$.
4. Solve the system and find x and y.

So reflection with respect to arbitrary lines can be found like this.

Example 2

Find the equation of the angle bisector of the lines $l_1 : 2x + 3y + 7 = 0$ and $l_2 : 3x + 2y - 11 = 0$.

Walkthrough

1. Let P be the intersection between l_1 and l_2.
2. Let Q be a point on the bisector.
3. Deduce an equation with the fact that the distance from Q to l_1 is same with the distance from Q to l_2.
4. Finish. There should be two lines.

Corollary : Equation of an Angle Bisector

Let $l_1 : a_1x + b_1y + c_1 = 0$ and $l_2 : a_2x + b_2y + c_2 = 0$ be two lines. Then the angle bisector of the two lines have equations

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}.$$

Example 3

Let ABC be a triangle with $A(3,5)$, $B(8,3)$, and $C(10,13)$. Find the coordinates of the centroid of $\triangle ABC$.

Walkthrough

1. Let D be the midpoint of BC. Find the coordinates of D.
2. Find the coordinates of G with the fact $\dfrac{AG}{GD} = 2$.

Corollary : Coordinate of a Centroid

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Then the centroid of $\triangle ABC$ has coordinates

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

Example 4

Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. If I is the incenter of $\triangle ABC$, then find BI.

Walkthrough

1. Let D be the intersection between the angle bisector of $\angle A$ and BC. Find the coordinates of D.
2. Similarly, find the coordinates of I.

Corollary : Coordinate of the Incenter

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Then the incenter of $\triangle ABC$ has coordinates

$$I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right),$$

where a, b, and c are the lengths of segments BC, CA, and AB, respectively.

Now, we will look at how to find an equation of a tangent line of a circle.

Example 5

Find the equation of the line passing $(-1,0)$ and tangent to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$.

Walkthrough

1. Let the slope of the line be m. Express the equation of the line in terms of m.
2. Use the point-line distance formula with the center.
3. Deduce an equation of m.

4 Coordinate Geometry Examples

Example 6 (AMC 12B 2013 P8)

Line l_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line l_2 has equation $y = 1$ and meets line l_1 at point B. Line l_3 has positive slope, goes through point A, and meets l_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of l_3?

Walkthrough

1. Find the coordinate of point B.
2. Let $C(x, y)$, and use the shoelace theorem to get an equation of x and y.
3. You should have two equations. Out of them, which makes l_3 have positive slope?

Example 7 (AMC 12B 2017 P9)

A circle has center $(-10, -4)$ and has radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

Walkthrough

1. Find the equations of two circles.
2. Find a way to extract a linear equation out of the two equations.

Example 8 (AMC 10A 2013 P18)

Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

Walkthrough

1. Find the area of the quadrilateral $ABCD$.
2. Let the intersection be E. What should the area of $\triangle ADE$ be?
3. Find the height of $\triangle ADE$.

Example 9 (AMC 12B 2017 P6)

The circle having $(0, 0)$ and $(8, 6)$ as the endpoints of a diameter intersects the x-axis at a second point. What is the x-coordinate of this point?

Walkthrough

1. Find the center of the circle.
2. Find the radius of the circle.
3. Find the equation of the circle.

5 General Examples

Example 10 (AMC 10B 2017 P15)

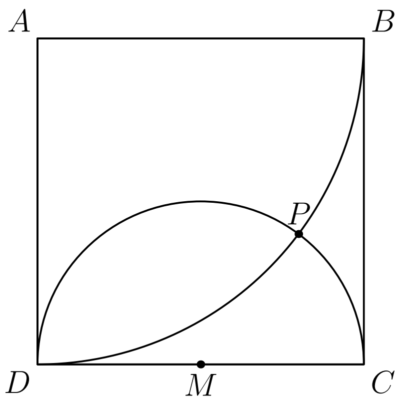
Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$?

Walkthrough

1. Set up coordinates, with D the origin, and $C(3,0)$.
2. Find the equation of the diagonal AC.

Example 11 (AMC 12A 2003 P17)

Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?



Walkthrough

1. Find the equation of the circle with radius 2.
2. Find the equation of the circle with radius 1.
3. Equate to find the coordinates of P.

Example 12 (AIME I 2013 P9)

A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. Find the length of the line segment along which the triangle is folded.

Walkthrough

1. Find the coordinates of A.
2. Find the coordinates of the folded A. Let this point be A'.
3. Find the equation of the folded line. Note that this line is the perpendicular bisector of AA'.
4. Find the coordinates of the endpoints.

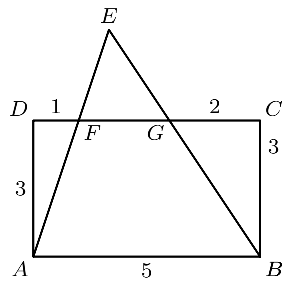
6 Problems

Problem 1 (AMC 12B 2021 P11)

Triangle ABC has $AB = 13, BC = 14$ and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

Problem 2 (AMC 12B 2003 P14)

In rectangle $ABCD, AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



Problem 3 (AMC 10A 2014 P18)

A square in the coordinate plane has vertices whose y-coordinates are 0, 1, 4, and 5. What is the area of the square?

Problem 4 (AMC 10A 2003 P22)

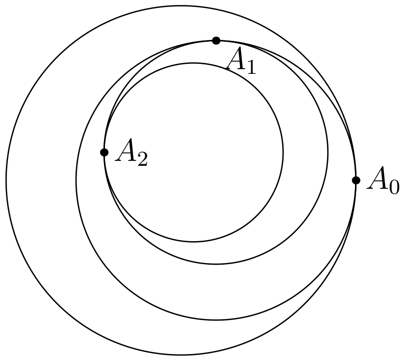
In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on BC with $BH = 6$, E is on AD with $DE = 4$, line EC intersects line AH at G, and F is on line AD with $GF \perp AF$. Find the length of GF.

Problem 5 (AIME I 2011 P3)

Let L be the line with slope $\frac{5}{12}$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.

Problem 6 (AIME II 2017 P12)

Circle C_0 has radius 1, and the point A_0 is a point on the circle. Circle C_1 has radius $r > 1$ and is internally tangent to C_0 at point A_0. Point A_1 lies on circle C_1 so that A_1 is located $90°$ counterclockwise from A_0 on C_1. Circle C_2 has radius r^2 and is internally tangent to C_1 at point A_1. In this way a sequence of circles $C_1, C_2, C_3, \ldots$ and a sequence of points on the circles $A_1, A_2, A_3, \ldots$ are constructed, where circle C_n has radius r^n and is internally tangent to circle C_{n-1} at point A_{n-1}, and point A_n lies on C_n $90°$ counterclockwise from point A_{n-1}, as shown in the figure below. There is one point B inside all of these circles. When $r = \frac{11}{60}$, the distance from the center C_0 to B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 7 (AIME I 2010 P13)

Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Problem 8 (AIME II 2014 P14)

In $\triangle ABC, AB = 10, \angle A = 30°$, and $\angle C = 45°$. Let H, D, and M be points on the line BC such that $AH \perp BC$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of the segment HM, and point P is on ray AD such that $PN \perp BC$. Then $AP^2 = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 9 (AIME I 2012 P13)

Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c, and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a+b+c+d$.

7 Solutions to Some Examples

Example 1

Find the point obtained by reflecting the point $(1, 2)$ with respect to the line $x + 2y - 3 = 0$.

Solution Note that $x + 2y - 3 = 0$ can be changed to $y = -\dfrac{1}{2}x + \dfrac{3}{2}$. Let the reflection be $P'(x, y)$.

First, since the midpoint of PP' is on the line $x + 2y - 3 = 0$, we have

$$\frac{1+x}{2} + 2 \cdot \frac{2+y}{2} - 3 = 0,$$

so $x + 2y - 1 = 0$.

Then, since segment PP' is perpendicular to $y = -\dfrac{1}{2} + \dfrac{3}{2}$. the slope of segment PP' is

$$\frac{y-2}{x-1} = 2,$$

and $2x - y = 0$. Solving the system

$$x + 2y - 1 = 0$$
$$2x - y = 0$$

gives $P'\left(\dfrac{1}{5}, \dfrac{2}{5}\right)$.

Example 2

Find the equation of the angle bisector of the lines $l_1 : 2x + 3y + 7 = 0$ and $l_2 : 3x + 2y - 11 = 0$.

Solution Let P be the intersection between l_1 and l_2. Take any point Q on the angle bisector, and let l_3 be the line perpendicular to the angle bisector PQ. Let A and B be the intersection between l_3 with l_1 and l_2, respectively.

Since we have $\triangle PAQ \cong \triangle PBQ$, $AQ = BQ$. Let $Q(x, y)$. Then

$$AQ = \frac{|2x + 3y + 7|}{\sqrt{2^2 + 3^2}}$$

$$BQ = \frac{|3x + 2y - 11|}{\sqrt{3^2 + 2^2}}.$$

By $AQ = BQ$, we have $|2x + 3y + 7| = |3x + 2y - 11|$. This gives two lines:

$$2x + 3y + 7 = 3x + 2y - 11$$

$$2x + 3y + 7 = -(3x + 2y - 11),$$

which are $x - y - 18 = 0$ and $5x + 5y - 4 = 0$.

> **Example 3**
>
> Let ABC be a triangle with $A(3, 5)$, $B(8, 3)$, and $C(10, 13)$. Find the coordinate of the centroid of $\triangle ABC$.

Solution Let D be the midpoint of BC. Then $D(9, 8)$. Since $\dfrac{AG}{GD} = 2$, we have

$$G\left(\frac{1 \cdot 3 + 2 \cdot 9}{1 + 2}, \frac{1 \cdot 5 + 2 \cdot 8}{1 + 2}\right) = (7, 7).$$

> **Example 4**
>
> Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. If I is the incenter of $\triangle ABC$, then find BI.

Solution Let $A(5, 12)$, $B(0, 0)$, and $C(14, 0)$. Let D be the intersection between the angle bisector of $\angle A$ and segment BC. Then since $\frac{BD}{DC} = \frac{AB}{AC} = \frac{13}{15}$, and $D(\frac{13}{2}, 0)$. Since I is the intersection between the angle bisector of $\angle B$ and segment AD, we have $\frac{AI}{DI} = \frac{AB}{BD} = \frac{13}{13/2} = 2$, so

$$I\left(\frac{1 \cdot 5 + 2 \cdot \frac{13}{2}}{1 + 2}, \frac{1 \cdot 12 + 2 \cdot 0}{1 + 2}\right) = (6, 4).$$

Therefore, $BI = 2\sqrt{13}$.

> **Example 5**
>
> Find the equation of the line passing $(-1, 0)$ and tangent to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$.

Solution The equation of the circle can be changed to $(x - 1)^2 + (y - 3)^2 = 4$. Let the line have slope m. Then the equation of the line will be $y = mx + m$, or $mx - y + m = 0$. Since this line is tangent to the circle, the distance between the line and point $(1, 3)$ will be 2, which is the radius. By the point-line distance formula, we have

$$\frac{|m \cdot 1 - 3 + m|}{\sqrt{m^2 + 1}} = 2,$$

so

$$\begin{aligned} |2m - 3| &= 2\sqrt{m^2 + 1} \\ (2m - 3)^2 &= 4(m^2 + 1) \\ 4m^2 - 12m + 9 &= 4m^2 + 4 \\ m &= \frac{5}{12}. \end{aligned}$$

Therefore, the equation of the line is $y = \frac{5}{12}x + \frac{5}{12}$, or $5x - 12y + 5 = 0$.