Intermediate Counting

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This handout is written assuming that the reader already knows the following topics of basic counting:

- Addition and Multiplication
- Permutations
- Combinations

1 Complementary Counting

Suppose you have the following problem:

> **Example 1**
> Find the number of two-digit numbers that the sum of the digits is at most 16.

Since the numbers range from 10 to 99, the sum of the digits range from 1 to 18. We can calculate

- Numbers whose digits sum to 1
- Numbers whose digits sum to 2
- ⋮
- Numbers whose digits sum to 16

and add them up to get the answer. This obviously takes a long time since we need to do 16 separate calculations. We try a different method.

Notice that since there are 90 two digit numbers in total, and their sum of digits are all between 1 and 18. So insteof looking for numbers whose digits sum to 1, 2, . . . , 16, we only look for numbers whose digits sum to 17 or 18. There are three numbers: 89, 98, and 99.

This is absolutely not what we wanted to find, but it is exactly *opposite* to what we wanted to find. Since there are a total of 90 numbers, the remaining 87 numbers have digits sum below 17, which is 1, 2, . . . , 16.

In set notation, in a universal set U, if we were to find $|A|$ (which denotes the number of elements of $|A|$), we use the identity

$$U = A \cup A^{\complement}$$

and instead find $|U|$ and $|A^{\complement}|$, since $|U| = |A| + |A^{\complement}|$.

Example 2

How many four digit positive integers have at least one even digit?

Example 3 (AMC 10A 2013 P7)

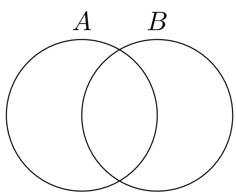
A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

Example 4

Suppose you are rearranging the letters of the word *equation*. How many rearrangements have at least one consonant either at the very front or at the very back? (The rearrangement does not have to be a valid word.)

2 Principle of Inclusion and Exclusion

Suppose you want to count the elements in $A \cup B$ where A and B are sets.



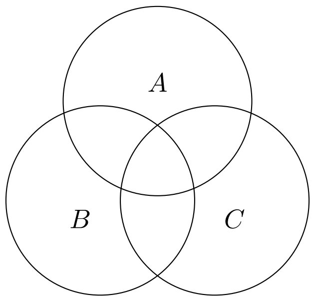
You can count the number of elements in three sets: $A - B$, $A \cap B$, and $B - A$. Adding up will give the number of elements in $A \cup B$.

This is already good, but we try a new method. We first count $|A|$ and $|B|$, and add up. This covers all of $A \cup B$, but the problem is that the elements in $A \cap B$ is counted twice! Once when counting $|A|$ and once again when counting $|B|$. Thus we need to subtract this. Therefore, we get

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This works with a slightly complicated formula to three sets. We have

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|.$$



Furthermore, the principle can be generalized into n sets.

Theorem : Principle of Inclusion and Exclusion

Let A_1, A_2, $\ldots$, A_i be sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Example 5

How many positive integers equal or less than 1001 is either a multiple of 7, multiple of 11, or a multiple of 13?

Example 6 (AMC 10B 2017 P13)

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Example 7 (AIME II 2017 P1)

Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

3 Symmetry

Many of you would have heard of the word *without loss of generality*. This arises from symmetry. This is best explained by an example.

Example 8

Suppose the digits from the five-digit number 12345 is rearranged. In how many numbers does 2 lie left to 3? One example of such number is 42351.

Note that the numbers 2 and 3 obviously cannot lie in the same position. So 2 should be either

- at the left of 3
- at the right of 3.

The number of numbers for the two cases are the same: if there is a number such that 2 lies at the left of 3, exchanging 2 and 3 will give a number such that 2 lies at the right of 3. For example, from 42351, changing the positions of 2 and 3 gives 43251. Since there is a one-to-one correspondence (bijection) between the numbers of two cases. the number of numbers for the two cases should be the same.

Since total 120 numbers are possible with rearranging, the desired number of numbers is $120/2 = 60$.

Simple.

Example 9

Suppose you are flipping a coin 12 times. How many configurations of the 12 flips has more heads than tails?

Example 10 (AMC 12A 2018 P13)

How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

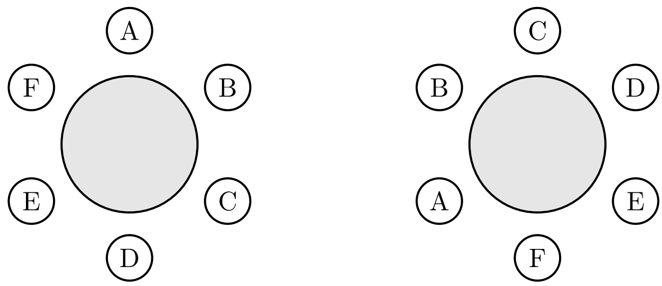
Note that the dividing factor may not always be two.

4 Circular Permutations

Suppose you have the following problem:

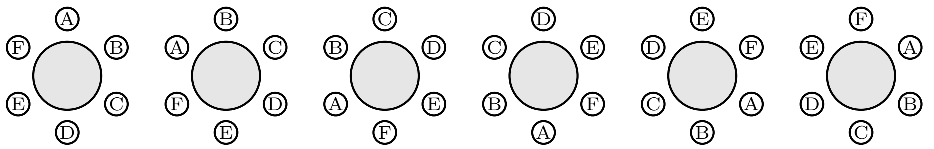
Example 11

There is a circular table with six seats, equally spaced. Suppose six people are sitting on the table. Two arrangements are identical if one arrangement can be formed by rotating the another, how many arrangements are possible?



So the two diagrams above are considered identical: rotating the first diagram 120° counterclockwise gives the second diagram.

If it was a normal permutation problem (without rotations), then there should be $6! = 720$ cases. How many overcounts are there?



Note that each arrangement is counted 6 times. Therefore, the number of possible arrangements is $6!/6 = 120$.

Theorem : Circular Permutations

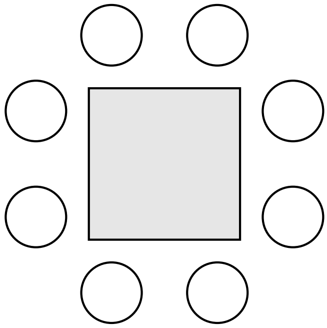
If n objects are arranged in a circle, the number of arrangements is $n!/n = (n-1)!$.

The problem can be approached as the following: first place one object anywhere. By rotation, it can be assumed that the object may be placed anywhere. This gives 1 case to place the first object. For the remaining $n-1$ objects, this is not a circular permutation anymore. Therefore, there are $(n-1)!$ cases to arrange these $n-1$ objects.

Some students just memorize the formula $(n-1)!$, but this is a very dangerous act. The main reason that the formula is $(n-1)!$ is that each configuration was counted n times. If there is a problem where each configuration is not counted n times, the formula won't work.

Example 12

Eight people are having a meal at a square-shaped table. Two people are sitting on one side of the square, each. If two arrangements are considered identical if one can be formed by rotating the another, how many arrangements are possible?



5 Stars and Bars

The technique of circles and bars is simply *combinations with repetition*.

Example 13

Find the number of ways of choosing three distinct one-digit numbers (including zero).

The number is obviously $\binom{10}{3} = 120$.

Example 14

Find the number of ways of choosing three one-digit numbers (including zero), allowing repetition.

This is a different problem: the previous example didn't allow triplets like $(1, 1, 8)$, but this example allows such triplets.

Solution We divide cases.

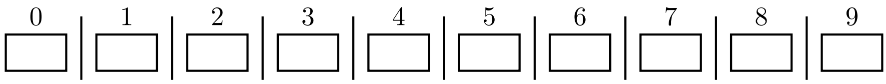
- If three numbers are distinct so that the numbers are of the form (a, b, c), then there are $\binom{10}{3} = 120$ ways.
- If two of the three numbers are the same so that the numbers are of the form (a, a, b), then there are $10 \cdot 9 = 90$ ways.

- If three numbers are all equal so that the numbers are of the form (a, a, a), then there are 10 cases.

Combining this gives $120 + 90 + 10 = 220$ cases.

This is a decent solution, but the solution will get complicated as numbers increase. We develop a new method called *stars and bars*.

Note that there are 10 choices of numbers and 3 chances to choose a number.



Suppose there are ten boxes separated by 9 bars, each labeled from 0 to 9. Let choosing number be the move of putting a star inside the box. In total, three stars will be in ten boxes (some boxes can be empty, and some boxes can have more than one star).

Consider the combination of 12 objects (9 bars and 3 stars). The number of arrangements of these 12 objects is $\binom{12}{3} = 220$. Notice that the arrangement of the 12 objects exactly corresponds to one arrangement of stars in boxes.

Therefore, the total number of triplets is $\binom{12}{3} = 220$.

This can be generalized. Suppose there are n objects, and you are choosing an object k times, allowing repetition. This will give n boxes separated by $n-1$ bars, and k stars to put in boxes. There are total $n+k-1$ objects, and the number of rearrangements is $\binom{n+k-1}{k}$.

Theorem : Stars and Bars

Suppose there are n objects, and you are choosing an object k times, allowing repetition. The number of possible k objects is

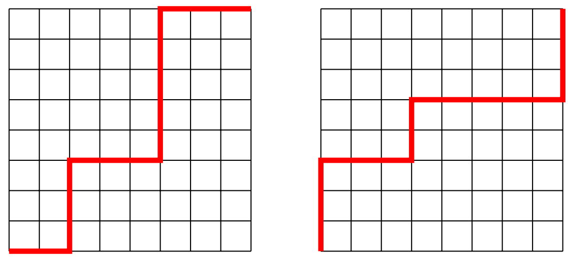
$$\binom{n+k-1}{k}.$$

Example 15 (AMC 10A 2003 P21)

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Example 16 (AIME I 2024 P6)

Consider the paths of length 16 that follow the lines from the lower left corner to the upper right corner on a 8×8 grid. Find the number of such paths that change direction exactly four times, like in the examples shown below.



Now, we look at a linear Diophantine equation.

Example 17

Find the number of 4-tuples (x, y, z, w) of nonnegative integers satisfying

$$x + y + z + w = 10.$$

This seems like a casework problem, not a stars and bars problem. However, here is the trick: first, distribute 10 to sum of 1s. That is,

$$10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

We will distribute the 1s to either x, y, z, or w. This looks familiar: there are 4 objects (which are x, y, z, and w) and you are going to choose one of them 10 times (allowing repetition). So for example, if you choose x three times, y two times, z five times, and w zero times, you get the 4-tuple $(3, 2, 5, 0)$.

Therefore, the number of 4-tuples is $\binom{4+10-1}{10} = 286$.

Remark.

Note that a variable may not be picked. That is, some variables can have a value of zero.

Example 18

Find the number of 4-tuples (x, y, z, w) of *positive* integers satisfying

$$x + y + z + w = 10.$$

Notice that this is different from the previous example! The previous example asked us for *nonnegative* integers, whereas this example asks us for *positive* integers. So a 4-tuple like $(3, 2, 5, 0)$ may not count.

How do we solve this, then? Since x, y, z, $w \geq 1$, we make the following substitutions:

$$x = x' + 1,\ y = y' + 1,\ z = z' + 1,\ w = w' + 1.$$

This will give

$$x' + y' + z' + w' = 6$$

with x', y', z', and $w' \geq 0$, which is the form we wanted. Thus, the number of possible 4-tuples is $\binom{4+6-1}{6} = 84$.

Example 19

Find the number of triplets (x, y, z) of nonnegative odd integers satisfying

$$x + y + z = 11.$$

Example 20

Find the number of triplets (x, y, z, w) of positive odd integers satisfying

$$x + y + z + 3w = 15.$$

Be careful of the coefficient 3 in front of w. How should we take care of that?

Example 21 (AIME I 2021 P4)

Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.