Laplace Transforms

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July 29, 2023

This handout covers about Laplace transforms that are used to solve ordinary differential equations. Laplace transform is a useful technique in solving ordinary differential equations. We look over definition, properties, and techniques for solving differential equations.

1 Definition of the Laplace Transform

Definition 1.1: Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

is said to be the **Laplace Transform** of f provided the integral converges.

We usually use the notation

$$\mathscr{L}\{f(t)\} = F(s),\ \mathscr{L}\{g(t)\} = G(s), \text{ and } \mathscr{L}\{y(t)\} = Y(s).$$

Example 1

Evaluate $\mathscr{L}\{1\}$.

Solution

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \left.\frac{-e^{-st}}{s}\right|_0^\infty = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

provided $s > 0$. If $s < 0$, the integral diverges.

Example 2

Evaluate $\mathscr{L}\{e^{at}\}$, where a is any real number.

Solution

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \left.\frac{e^{(-s+a)t}}{-s+a}\right|_0^\infty = \lim_{b\to\infty}\frac{e^{(-s+a)b}-1}{-s+a} = \frac{1}{s-a}$$

provided $s > a$. If $s < a$, the integral diverges.

From now on, we use the notation $\int_0^\infty f(t)\,dt$ as $\lim_{b\to\infty}\int_0^b f(t)\,dt$. Also, we assume that the conditions for s are satisfied.

Theorem 1.1: Linearity of the Laplace Transform

Suppose that there exists $\mathscr{L}\{f_1\}$ and $\mathscr{L}\{f_2\}$ for $s > a_1$ and $s > a_2$. Then, for $s > max\{a_1, a_2\}$,

$$\mathscr{L}\{c_1f_1 + c_2f_2\} = c_1\mathscr{L}\{f_1(t)\} + c_2\mathscr{L}\{f_2(t)\}.$$

Proof.

$$\begin{aligned}\mathscr{L}\{c_1f_1 + c_2f_2\} &= \int_0^\infty e^{-st}(c_1f_1(t) + c_2f_2(t))\,dt \\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt \\ &= c_1\mathscr{L}\{f_1(t)\} + c_2\mathscr{L}\{f_2(t)\}.\end{aligned}$$

■

Some transforms of basic functions are:

$$\mathscr{L}\{1\} = \frac{1}{s}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1, 2, 3\dots, \quad \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathscr{L}\{\sin\ kt\} = \frac{k}{s^2+k^2}, \quad \mathscr{L}\{\cos\ kt\} = \frac{s}{s^2+k^2}$$

Example 3

Evaluate $\mathscr{L}\{t - t^2 + 2e^{4t}\}$.

Solution

$$\mathscr{L}\{t - t^2 + 2e^{4t}\} = \mathscr{L}\{t\} - \mathscr{L}\{t^2\} + 2\mathscr{L}\{e^{4t}\} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s-4}.$$

Of course, the improper integral $\int_0^\infty f(t)e^{st}\,dt$ might not exist. Then, when does the Laplace transform exist? We propose a theorem of a condition for existence. We first define two terminologies, *piecewise continuous* and *exponential order*.

Definition 1.2: Piecewise Continuous Function

A function f is piecewise continuous when the number of discontinuous points in $(-\infty, \infty)$ are finite.

Definition 1.3: Exponential Order

A function f is of exponential order when there exists constants a, $k > 0$ and $T > 0$ such that

$$f(t) \leq ke^{aT}.$$

This means that f should be eventually smaller than an exponential function. For example, $f(t) = t^n$ is of exponential order for any natural number n, but $f(t) = e^{t^2}$ is not of exponential order.

Theorem 1.2: Sufficient Condition for the Existence of Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order. Then the Laplace transform of f exists for $s > 0$.

Proof. We divide $[0, \infty)$ to $[0, T)$ and $[T, \infty)$.

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt.$$

We get that $\int_0^T e^{-st} f(t)\ dt$ is finite. Since f is of exponential order, there exists some constants a, $k > 0$ and $T > 0$ such that

$$|f(t)| \leq Me^{ct} \text{ for } t > T.$$

Therefore,

$$\begin{aligned}\left| \int_T^\infty e^{-st} f(t)\ dt \right| &\leq \int_T^\infty |e^{-st} f(t)|\ dt \\ &\leq M \cdot \int_0^\infty e^{-st} \cdot e^{ct}\ dt \\ &= M \cdot \frac{e^{-(s-c)T}}{s-c} \text{ for } s > c.\end{aligned}$$

■

We now know about existence, but how about uniqueness? What if there are two different Laplace transforms for a function? That is actually not the case, and Laplace transform is unique. However, the proof of uniqueness is beyond this level, so we do not state here. From now on, one can assume that Laplace transforms of functions are unique.

Theorem 1.3: Uniqueness of the Laplace Transform

Assume that f, $g : [0,\infty) \to \mathbf{R}$ are continuous and of exponential order. If $\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\}$, the $f(t) = g(t)$.

2 The Inverse Laplace Transform

Definition 2.1: Inverse Laplace Transform

If $F(s) = \mathscr{L}\{f(t)\}$ we say that $f(t)$ is the **Inverse Laplace Transform** of $F(s)$.

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Inverse transforms of some functions are:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n,\ n = 1,2,3\ldots, \quad \mathscr{L}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathscr{L}\left\{\frac{k}{s^2+k^2}\right\} = \sin\ kt, \quad \mathscr{L}\left\{\frac{s}{s^2+k^2}\right\} = \cos\ kt$$

Like the Laplace transform, the inverse transform is also linear.

Theorem 2.1: Linearity of the Inverse Transform

The inverse Laplace transform is a linear transform. That is, for constants c_1 and c_2,

$$\mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\}.$$

Example 4

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3.$$

Example 5

Evaluate $\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+9}\right\}$.

Solution

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+9}\right\} &= \mathscr{L}^{-1}\left\{2\cdot\frac{s}{s^2+9}+\frac{3}{s^2+9}\right\}\\ &= 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\\ &= 2\cos 3t+\sin 3t.\end{aligned}$$

3 Transforms of Derivatives and Integrals

In this section, we see some properties of Laplace transforms and how they can be used to solve ordinary differential equations.

Theorem 3.1: Transforms of Derivatives

If f' is continuous on $[0,\infty)$ and assume that f is of exponential order. Then,

$$\mathscr{L}\{f'(t)\} = sF(s)-f(0).$$

If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0).$$

Proof. We use induction. For $n=1$,

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty e^{-st}f'(t)\,dt\\ &= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt\\ &= -f(0)+s\mathscr{L}\{f(t)\}\\ &= sF(s)-f(0).\end{aligned}$$

Assume the equation holds for $n=k$. So,

$$\mathscr{L}\{f^{(k)}(t)\} = s^kF(s)-s^{k-1}f(0)-s^{k-2}f'(0)-\cdots-f^{(k-1)}(0).$$

For $n = k + 1$,

$$\begin{aligned}\mathscr{L}\{f^{(k+1)}(t)\} &= \int_0^\infty e^{-st} f^{(k+1)}(t)\, dt \\ &= e^{-st} f^{(k)}(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(k)}(t)\, dt \\ &= -f^{(k)}(0) + s\mathscr{L}\{f^{(k)}(t)\} \\ &= s(s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \cdots - f^{(k-1)}(0)) - f^{(k)}(0) \\ &= s^{k+1} F(s) - s^k f(0) - s^{k-1} f'(0) - \cdots - f^{(k)}(0),\end{aligned}$$

which completes the induction. ■

Theorem 3.2: Transforms of Integrals

If f is piecewise continuous on $[0, \infty)$ and assume that f is of exponential order. Then,

$$\mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}, \text{ and}$$

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau.$$

Proof. Let $g(t) = \int_0^t f(\tau)\, d\tau$. We first prove that $g(t)$ is of exponential order. Since f is of exponential order, there exists k, a and τ such that $|f(t)| \leq ke^{at}$. Then,

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right| \leq \int_0^t |f(\tau)|\, d\tau \leq \int_0^t ke^{a\tau}\, d\tau = \frac{k}{a}(e^{at} - 1) < \frac{k}{a}e^{at},$$

which shows that g is also of exponential order. Also, since $\frac{d}{dt}g(t) = f(t)$, and $g(0) = 0$, by the Transforms of Derivatives theorem,

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{d}{dt}g(t)\right\} = s\mathscr{L}\{g(t)\}(s) - g(0) = s\mathscr{L}\{g(t)\}.$$

Dividing by s for both sides gives us

$$\mathscr{L}\{g(t)\} = \mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}.$$ ■

Example 6

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}$$
$$= \int_0^t sin\tau \, d\tau$$
$$= 1 - cost.$$

Example 7

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+1)}\right\}$$
$$= \int_0^t (1 - cos\tau) \, d\tau$$
$$= t - sint.$$

Solving Differential Equations with Laplace Transforms

Laplace transforms can be used in solving ordinary differential equations, especially initial-value problems. The steps for solving initial-value problems are:

1. Apply the Laplace transform for both sides of the initial-value problem.
2. Solve the equation with respect to $F(s)$.
3. Apply the inverse transform to the solution of $F(s)$, and you get the solution $f(t)$ to the initial-value problem.

Example 8

Solve $y' + y = 2cost$, $y(0) = 1$.

Solution Applying Laplace transform to both sides gives you

$$sY(s) - y(0) + Y(s) = 2\cdot\frac{s}{s^2+1}$$
$$(s+1)Y(s) - 1 = 2\cdot\frac{s}{s^2+1}.$$

If you solve for $Y(s)$, you get

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{2s}{(s+1)(s^2+1)} \\ &= \frac{s}{s^2+1} + \frac{1}{s^2+1}. \end{aligned}$$

If you apply the inverse transform for both sides, you finally obtain

$$y(t) = \cos t + \sin t$$

which is the solution of the equation given.

4 Translation Theorems

Theorem 4.1: First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\begin{aligned} \mathscr{L}\{e^{at} f(t)\} &= F(s-a), \text{and} \\ \mathscr{L}^{-1}\{F(s-a)\} &= e^{at} f(t). \end{aligned}$$

We also use the notation $F(s)\Big|_{s \to s-a}$ for $F(s-a)$.

Proof.

$$\begin{aligned} \mathscr{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-(s-a)t} f(t)\ dt \\ &= F(s-a). \end{aligned}$$ ■

Example 9

Evaluate $\mathscr{L}\{e^{2t} t^5\}$.

Solution

$$\mathscr{L}\{e^{2t} t^5\} = \mathscr{L}\{t^5\}_{s \to s-2} = \frac{5!}{s^6}\Bigg|_{s \to s-2} = \frac{120}{(s-2)^6}.$$

Example 10

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - 4s + 13}\right\}$.

Solution

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{s^2-4s+13}\right\} &= \mathscr{L}^{-1}\left\{\left.\frac{s+2}{s^2+9}\right|_{s\to s-2}\right\} \\ &= \mathscr{L}^{-1}\left\{\left.\left(\frac{s}{s^2+9}+\frac{2}{3}\cdot\frac{3}{s^2+9}\right)\right|_{s\to s-2}\right\} \\ &= e^{2t}\cos 3t+\frac{2}{3}e^{2t}\sin 3t.\end{aligned}$$

> **Example 11**
> Solve $y''-2y'+1y=te^t$, $y(0)=0$, $y'(0)=4$.

Solution Applying Laplace transform to both sides gives you

$$\begin{aligned}\mathscr{L}\{y''\}-2\mathscr{L}\{y'\}+\mathscr{L}\{y\} &= \mathscr{L}\{te^t\} \\ s^2Y(s)-sy(0)-y'(0)-2Y(s)+2y(0)+Y(s) &= \frac{1}{(s-1)^2}\end{aligned}$$

If you solve for $Y(s)$, you get

$$\begin{aligned}(s^2-2s+1)Y(s) &= 4+\frac{1}{(s-1)^2} \\ Y(s) &= \frac{4}{(s-1)^2}+\frac{1}{(s-1)^4}.\end{aligned}$$

If you apply the inverse transform for both sides, you finally obtain

$$y(t)=4te^t+\frac{1}{6}t^3e^t.$$

Definition 4.1: Unit Step Function

The **unit step function** $\mathscr{U}(t-a)$ is defined as

$$\mathscr{U}(t-a)=\begin{cases}0 & t<a\\ 1 & t\geq a.\end{cases}$$

People also call this function as the *heaviside function*. However, we will call it as unit step function here.

When a function is multiplied by $\mathscr{U}(t-a)$, the function becomes 0 for $t<a$, and itself for $t\geq a$. That is,

$$f(t)\mathscr{U}(t-a)=\begin{cases}0 & t<a\\ f(t) & t\geq a.\end{cases}$$

If you want to shift the function a units to the right, you can take

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a. \end{cases}$$

Also, general piecewise functions of the type

$$f(t) = \begin{cases} g(t) & t < a \\ h(t) & t \geq a. \end{cases}$$

can be expressed as

$$f(t) = g(t) - \big(g(t) - h(t)\big)\mathscr{U}(t-a).$$

Similarly, piecewise functions of three cases

$$f(t) = \begin{cases} g(t) & t < a \\ h(t) & a \leq t < b \\ g(t) & t \geq b \end{cases}$$

can be written

$$f(t) = g(t) + \big(h(t) - g(t)\big)\big[\mathscr{U}(t-a) - \mathscr{U}(t-b)\big].$$

One can generalize this to functions of several cases, even more than three.

Theorem 4.2: Second Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s), \text{ and}$$
$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

This can also be written as

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

when you put $g(t) = f(t-a)$.

Proof.

$$\begin{aligned}\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} &= \int_0^a e^{-st} f(t-a)\mathscr{U}(t-a)\ dt + \int_a^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_0^a 0\ dt + \int_a^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_a^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\ dt.\end{aligned}$$

If we substitute $v = t-a$, since $dv = dt$,

$$\begin{aligned}\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} &= \int_a^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_a^\infty e^{-s(v+a)} f(v)\ dv \\ &= e^{-as}\int_a^\infty e^{-sv} f(v)\ dv \\ &= e^{-as}\mathscr{L}\{f(t)\}.\end{aligned}$$

■

Corollary : Laplace Transform of a Unit Step Function

$$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\left\{\frac{1}{s}\right\} = \frac{e^{-as}}{s}$$

Example 12

Evaluate $\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\}$.

Solution

$$\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\cos(t+\pi)\} = -e^{\pi s}\mathscr{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}.$$

Example 13

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s-2}e^{-6s}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s-2}e^{-6s}\right\} = e^{2(t-6)}\mathscr{U}(t-6).$$

Example 14

Solve $y' - 2y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin t & t \ge \pi. \end{cases}$

Solution $f(t)$ can be written as $f(t) = \sin t\mathscr{U}(t-\pi)$.

Applying Laplace transform to both sides gives you

$$\mathscr{L}\{y'\} - 2\mathscr{L}\{y'\} = \mathscr{L}\{\sin t\mathscr{U}(t-\pi)\}$$
$$sY(s) - y(0) - 2Y(s) = \frac{1}{s^2+1}e^{-\pi s}$$

If you solve for $Y(s)$, you get

$$(s-2)Y(s) = -\frac{1}{s^2+1}e^{-\pi s}$$
$$Y(s) = -\frac{1}{(s-2)(s^2+1)}$$
$$= \frac{1}{5}\cdot\frac{s+2}{s^2+1} - \frac{1}{5}\cdot\frac{1}{s-2}$$

If you apply the inverse transform for both sides, you finally obtain

$$y(t) = \frac{1}{5}\cos(t-\pi)\mathscr{U}(t-\pi) + \frac{2}{5}\sin(t-\pi)\mathscr{U}(t-\pi) - \frac{1}{5}e^{2(t-\pi)}\mathscr{U}(t-\pi)$$
$$= \begin{cases} 0 & t < \pi \\ \frac{1}{5}\cos(t-\pi) + \frac{2}{5}\sin(t-\pi) - \frac{1}{5}e^{2(t-\pi)} & t \geq \pi. \end{cases}$$

The solution of a differential equation including unit step functions may not be differentiable at some points. In this case, we differentiate piecewise, so that the function is continuous, and each part of the function satisfies the differential equation. For the example above, each side of the solution satisfies the differential equation. Also, the solution is continuous because $\lim_{t\to\pi} y(t) = 0 = y(\pi)$.

5 Derivatives and Integrals of Transforms

Theorem 5.1: Derivatives of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, \cdots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s), \text{ and}$$

$$\mathscr{L}^{-1}\left\{\frac{d^n}{ds^n}\right\} = (-1)^n t^n f(t).$$

Proof. We use induction. For $n = 1$, since

$$\begin{aligned}
\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\ dt \\
&= \int_0^\infty \frac{\partial}{\partial s}\big(e^{-st}f(t)\big)\ dt \text{ (by Leibniz Rule)} \\
&= \int_0^\infty -e^{-st}\cdot tf(t)\ dt = -\mathscr{L}\{tf(t)\}, \text{ and}
\end{aligned}$$

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}.$$

Assume the equation holds for $n = k$. So,

$$\mathscr{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k}F(s).$$

For $n = k + 1$,

$$\begin{aligned}
\frac{d}{ds}\left((-1)^k \frac{d^k}{ds^k}F(s)\right) &= (-1)^k \frac{d^{k+1}}{ds^{k+1}}F(s) \\
&= \frac{d}{ds}\int_0^\infty e^{-st}t^k f(t)\ dt \\
&= \int_0^\infty \frac{\partial}{\partial s}\big(e^{-st}t^k f(t)\big)\ dt \text{ (by Leibniz Rule)} \\
&= \int_0^\infty -e^{-st}\cdot t\cdot t^k f(t)\ dt = -\mathscr{L}\{t^{k+1}f(t)\}.
\end{aligned}$$

Therefore,

$$\mathscr{L}\{t^{k+1}f(t)\} = (-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}},$$

which completes the induction. ■

The Leibniz Rule used in the proof is a theorem that interchanges the derivative operator with the partial derivative operator inside the integral.

$$\frac{d}{ds}\left(\int_a^b f(x,t)\ dt\right) = \int_a^b \frac{\partial}{\partial x}f(x,t)\ dt.$$

Theorem 5.2: Integrals of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(r)\,dr, \text{ and}$$

$$\mathscr{L}^{-1}\left\{\int_s^\infty F(r)\,dr\right\} = \frac{f(t)}{t}.$$

Proof.

$$\begin{aligned}\int_s^\infty F(r)\,dr &= \int_s^\infty \left(\int_0^\infty e^{-rt} f(t)\,dt\right) dr \\ &= \int_0^\infty \left(\int_s^\infty e^{-rt} f(t)\,dr\right) dt \text{ (Changing the order of integration)} \\ &= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt \\ &= \int_0^\infty e^{-st}\frac{f(t)}{t} \\ &= \mathscr{L}\left\{\frac{f(t)}{t}\right\}.\end{aligned}$$

■

Example 15

Evaluate $\mathscr{L}^{-1}\left\{\ln\frac{s+3}{s-2}\right\}$.

Solution

$$\begin{aligned}\text{Since } \frac{d}{ds}\left(\ln\frac{s+3}{s-2}\right) &= \frac{d}{ds}\big(\ln(s+3) - \ln(s-2)\big) \\ &= \frac{1}{s+3} - \frac{1}{s-2} = \mathscr{L}\{-tf(t)\},\end{aligned}$$

$$\begin{aligned}-tf(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s+3} - \frac{1}{s-2}\right\} \\ &= e^{-3t} - e^{2t},\end{aligned}$$

$$\text{and } f(t) = \frac{e^{2t} - e^{-3t}}{t}.$$

Example 16

Solve $y'' + y = te^t$, $y(0) = 0$, $y'(0) = 1$.

Solution Apply Laplace transforms to both sides, we get

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = \mathscr{L}\{te^t\}$$
$$s^2Y(s) - sy(0) - y'(0) + Y(s) = -\frac{d}{ds}\frac{1}{s-1}$$

Solve for $Y(s)$, then

$$\begin{aligned}(s^2+1)Y(s) &= 1 + \frac{1}{(s-1)^2} \\ Y(s) &= \frac{1}{s^2+1} + \frac{1}{(s^2+1)(s-1)^2} \\ &= \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{1}{2}\frac{1}{s-1}\end{aligned}$$

Finally, applying inverse transform to both sides gives you the solution

$$y(t) = \frac{1}{2}\cos t + \sin t + \frac{1}{2}te^t - \frac{1}{2}e^t.$$

6 Convolution

Definition 6.1: Convolution

If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the **convolution** of f and g, denoted $f * g$, is a function defined by

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Example 17

Evaluate $t * \sin t$.

Solution

$$\begin{aligned} e^t * t &= \int_0^t e^\tau \cdot (t-\tau)\, d\tau \\ &= \int_0^t \left(te^\tau - \tau e^\tau\right) d\tau \\ &= te^t - t - te^t + e^t - 1 = e^t - t - 1. \end{aligned}$$

Then, when is convolution used and why is it defined like this? The reason is that it makes one able to multiply two transforms together! The theorem is called the *convolution theorem*.

Theorem 6.1: Convolution Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s), \text{ and}$$
$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g.$$

Proof.

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \text{ and } G(s) = \mathscr{L}\{g(t)\} = \int_0^\infty e^{-s\gamma} g(\gamma)\, d\gamma.$$

Then,

$$\begin{aligned} F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right)\left(\int_0^\infty e^{-s\gamma} g(\gamma)\, d\gamma\right) \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+\gamma)} f(\tau) g(\gamma)\, d\tau\, d\gamma \\ &= \int_0^\infty f(\tau)\, d\tau \int_0^\infty e^{-s(\tau+\gamma)} g(\gamma)\, d\gamma \end{aligned}$$

If we let $t = \tau + \gamma$, since $dt = d\gamma$, so

$$F(s)G(s) = \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty e^{-st} g(t-\tau)\, dt.$$

Because f and g are piecewise continuous on $[0, \infty)$ and of exponential order, we can change the order of integration. Therefore,

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-st}\,dt \int_0^t f(\tau)g(t-\tau)\,d\tau \\ &= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt \\ &= \mathscr{L}\{f * g\}. \end{aligned}$$

■

Corollary : Transforms of Integrals

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}.$$

Proof.

$$\begin{aligned} \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} &= \mathscr{L}\left\{\int_0^t f(\tau)\cdot 1\,d\tau\right\} \\ &= \mathscr{L}\{f(t) * 1\} \\ &= \mathscr{L}\{f(t)\}\cdot\mathscr{L}\{1\} \\ &= F(s)\cdot\frac{1}{s} \\ &= \frac{F(s)}{s}. \end{aligned}$$

■

Example 18

Evaluate $\mathscr{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\}$.

Solution

$$\begin{aligned} \mathscr{L}\left\{\int_0^t \cos\tau\sin(t-\tau)\,d\tau\right\} &= \mathscr{L}\{\cos t * \sin t\} \\ &= \mathscr{L}\{\cos t\}\cdot\mathscr{L}\{\sin t\} \\ &= \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}. \end{aligned}$$

Example 19

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$.

Solution

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\cdot\frac{1}{s^2+k^2}\right\} \\
&= \frac{1}{k^2}\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\cdot\frac{k}{s^2+k^2}\right\} \\
&= \frac{1}{k^2}(\sin t * \sin t) \\
&= \frac{1}{k^2}\int_0^t \sin k\tau \;\; \sin k(t-\tau)\ d\tau \\
&= \frac{1}{k^2}\int_0^t \frac{1}{2}\big(\cos k(2\tau-t) - \cos kt\big)\ d\tau \\
&= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau-t) - \tau\cos kt\right]_0^t \\
&= \frac{sinkt - ktcoskt}{2k^3}.
\end{aligned}$$

Properties of Convolution

Convolution has the following properties:

- The associative property, i.e. $f*(g*h) = (f*g)*h$
- The commutative property, i.e. $f*g = g*f$
- The distributive property, i.e. $f*(g+h) = f*g + f*h$
- $f*0 = 0*f = 0$.

Integral Equations

There are not only differential equations, but also integral equations! *Integral equations* are simply equations that contain integrals. Solving integral equations are very similar to solving differential equations, using Laplace transforms. Especially, the convolution theorem is used frequently while solving integral equations. There are also equations that contain both derivatives and integrals. Such equations are called *integrodifferential equations*.

Example 20

Solve $y(t) + \int_0^t y(\tau)e^{t-\tau} = 3t^2$.

Solution First, we apply the Laplace transform for both sides.

$$\begin{aligned}\mathscr{L}\{y(t)\} + \mathscr{L}\left\{\int_0^t y(\tau)e^{t-\tau}\right\} &= \mathscr{L}\{3t^2\} \\ \mathscr{L}\{y(t)\} + \mathscr{L}\{y(t) * e^t\} &= \mathscr{L}\{3t^2\} \\ \mathscr{L}\{y(t)\} + \mathscr{L}\{y(t)\} \cdot \mathscr{L}\{e^t\} &= \mathscr{L}\{3t^2\} \\ Y(s) + \frac{1}{s-1}Y(s) &= \frac{6}{s^3}\end{aligned}$$

Then, solving for $Y(s)$ gives

$$\begin{aligned}\frac{s}{s-1}Y(s) &= \frac{6}{s^3} \\ Y(s) &= \frac{6s-6}{s^4} \\ &= \frac{6}{s^3} - \frac{6}{s^4} = 3\frac{2}{s^3} - \frac{6}{s^4}.\end{aligned}$$

Therefore, if you apply the inverse transform, you get the solution

$$y(t) = 3t^2 - t^3.$$

7 The Dirac Delta Function

Definition 7.1: Unit Impulse

The **unit impulse function** $\delta_a(t-t_0)$ is defined as

$$\delta_a(t-t_0) = \begin{cases} 0 & t < t_0 - a \\ \frac{1}{2a} & t_0 - a \le t < t_0 + a \\ 0 & t \ge t_0 + a \end{cases}$$

where $a > 0$ and $t_0 > 0$.

The unit impulse function has the following property:

$$\int_0^\infty \delta_a(t-t_0) = 1.$$

Definition 7.2: Dirac Delta Function

The **Dirac delta function** $\delta(t-t_0)$ is defined by the limit

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0).$$

The Dirac delta function has the following properties:

- $\delta(t-t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0, \end{cases}$ and
- $\int_0^\infty \delta(t-t_0)\ dt = 1.$

For usual functions, $\int_0^\infty \delta(t-t_0)\ dt = 0$, but actually $\int_0^\infty \delta(t-t_0)\ dt = 1$. The Dirac delta function is not actually a function–it is a distribution. The Dirac delta function doesn't contain any meaning itself, but it is characterized with other functions during integration.

Theorem 7.1: Shifting Property of Dirac Delta Function

If f is a continuous function, then

$$\int_0^\infty \delta(t-t_0)f(t)\ dt = f(t_0).$$

Proof.

$$\begin{aligned}\int_0^\infty \delta(t-t_0)f(t)\ dt &= \lim_{a\to 0}\int_0^\infty \delta_a(t-t_0)f(t)\ dt \\ &= \lim_{a\to 0}\frac{1}{2a}\int_{t_0-a}^{t_0+a} f(t)\ dt\end{aligned}$$

By the mean value theorem for integrals, there exists $\tilde{t} \in (t_0-a, t_0+a)$ such that

$$\int_{t_0-a}^{t_0+a} f(t)\ dt = 2af(\tilde{t}).$$

Finally,

$$\begin{aligned}\int_0^\infty \delta(t-t_0)f(t)\ dt &= \lim_{a\to 0}\frac{1}{2a}\int_{t_0-a}^{t_0+a} f(t)\ dt \\ &= \lim_{a\to 0}\frac{1}{2a}\left(2af(\tilde{t})\right) \\ &= f(t_0)\end{aligned}$$

Since $\tilde{t}\to 0$ as $a\to\infty$. ■

Theorem 7.2: Transform of the Dirac Delta Function

For $t_0 > 0$,

$$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}.$$

There are two proofs, using the shifting property or the unit step function. We state both.

Proof. If we set $f(t) = e^{-st}$, then

$$\mathscr{L}\{\delta(t-t_0)\} = \int_0^\infty \delta(t-t_0)\cdot e^{-st}\ dt = e^{-st_0}$$

Since $f(t_0) = e^{-st_0}$. ■

Proof. We first write the Dirac delta function as a combination of unit step functions.

$$\delta_a(t-t_0) = \frac{1}{2a}\Big(\mathscr{U}\left(t-(t_0-a)\right) - \mathscr{U}\left(t-(t_0+a)\right)\Big).$$

If we apply the Laplace transform,

$$\begin{aligned}\mathscr{L}\{\delta_a(t-t_0)\} &= \mathscr{L}\left\{\frac{1}{2a}\Big(\mathscr{U}\left(t-(t_0-a)\right) - \mathscr{U}\left(t-(t_0+a)\right)\Big)\right\} \\ &= \frac{1}{2a}\left(\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right) \\ &= e^{-st_0}\left(\frac{e^{as}-e^{-as}}{2as}\right).\end{aligned}$$

Since the Dirac delta function is the unit impulse when $a \to 0$,

$$\begin{aligned}\mathscr{L}\{\delta(t-t_0)\} &= \lim_{a\to 0}\mathscr{L}\{\delta_a(t-t_0)\}\\ &= e^{-st_0}\lim_{a\to 0}\left(\frac{e^{as}-e^{-as}}{2as}\right)\\ &= e^{-st_0}\lim_{a\to 0}\left(\frac{se^{as}+se^{-as}}{2as}\right) \text{ (by L'Hôpital's Rule)}\\ &= e^{-st_0}.\end{aligned}$$

■

Corollary

$\mathscr{L}\{\delta(t-0)\} = 1.$

Solving differential equations containing the Dirac delta function is similar with those without the Dirac delta function. The Dirac delta function comes out when one generates a differential equation with a function that is not differentiable at some point. The Dirac delta function in a differential equation doesn't contain a meaning itself, and something comes up only when one applies the Laplace transform. Since an exponential function comes out when you apply Laplace transform of the Dirac-delta function, the solution of the differential equation containing the Dirac-delta function contains unit-step functions.

Example 21

Solve $y'' + y = \delta(t-\pi)$, $y(0) = -2$, and $y'(0) = 0$.

Solution If you apply the Laplace transform for both sides, you get

$$s^2Y(s) + 2s + Y(s) - 0 = e^{-\pi s}.$$

Then, solving for $Y(s)$ gives you

$$\begin{aligned}(s^2+1)Y(s) &= -2s + e^{-\pi s}\\ Y(s) &= -2\cdot\frac{s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\end{aligned}$$

Using inverse transform theorem, you get

$$\begin{aligned}y(t) &= -2 + \sin(t-\pi)\mathscr{U}(t-\pi)\\ &= \begin{cases}-2 & t < 2\pi\\ -2- & t \geq \pi.\end{cases}\end{aligned}$$

8 Systems of Differential Equations

Laplace transform can also be applied when solving systems of differential equations, mostly linear systems. After applying the Laplace transform, one can solve the system of algebraic equations, and then apply the inverse theorem to get the solution of the system.

Example 22

Solve

$$x' + y = \cos 2t$$
$$-x + y' = \sin 2t.$$

when $x(0) = 0$ and $y(0) = 0$.

Solution Applying Laplace transform for both equations, you obtain the system of equations

$$sX(s) - 0 + Y(s) = \frac{s}{s^2+4}$$
$$-X(s) + sY(s) - 0 = \frac{2}{s^2+4}$$

which is the same as

$$sX(s) + Y(s) = \frac{s}{s^2+4}$$
$$-X(s) + sY(s) = \frac{2}{s^2+4}.$$

Solving the system of algebraic equations of $X(s)$ and $Y(s)$ yields

$$X(s) = \frac{s^2-2}{(s^2+1)(s^2+4)} = -\frac{1}{s^2+1} + \frac{2}{s^2+4}$$
$$Y(s) = \frac{3s}{(s^2+1)(s^2+4)} = \frac{s}{s^2+1} - \frac{s}{s^2+4}.$$

Therefore, the solution is

$$x(t) = -\sin t + \sin 2t$$
$$y(t) = \cos t - \cos 2t.$$

9 Exercises

Orange problems are calculation problems, while blue problems are miscellaneous problems.

Exercise 1 (Transform of Trigonometric Functions)

Prove that $\mathscr{L}\{\sin \alpha t\} = \dfrac{\alpha}{s^2+\alpha^2}$ and $\mathscr{L}\{\cos \alpha t\} = \dfrac{s}{s^2+\alpha^2}$ by using **Euler's Formula** $e^{ix} = \cos x + i \sin x$.

Exercise 2 (Not Exponential Order Function)

Explain why $f(t) = e^{t^2}$ is not of exponential order.

Exercise 3 (Two Functions of the Same Transform)

In section 1, we stated that the Laplace transform $\mathscr{L}\{f(t)\}$ is unique if $f(t)$ is continuous on $[0, \infty)$. Find two functions f and g, not necessarily continuous, such that $\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\}$.

Exercise 4 (Transform of t^a where $\alpha \notin \mathbf{N}$)

The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

for $\alpha > 0$. Use this formula to find $\mathscr{L}\{t^\alpha\}$ where $\alpha > -1$ is any real number.

Exercise 5 (Transform of a Taylor Series)

Recall that the **Taylor Series** of a function is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Show that $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ by expanding e^{at} as a Taylor series and using the formula $\mathscr{L}\{t^n\} = n!/s^{n+1}$.

Exercise 6

Solve $y'' + 4y = 5e^t - 10e^{-t}$, $y(0) = 2$, $y'(0) = 7$.

Exercise 7 (End Behavior of $F(s)$)

If the Laplace transform of $f(t)$ exists, prove that $\lim_{s\to\infty} \mathscr{L}\{f(t)\} = 0$.

Exercise 8 (Initial Value Theorem)

If $\mathscr{L}\{f(t)\} = F(s)$, prove that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Exercise 9 (Final Value Theorem)

If $\mathscr{L}\{f(t)\} = F(s)$, prove that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Exercise 10

Solve $y'' - 8y' + 16y = t^2 e^{4t}$, $y(0) = 0$, $y'(0) = 2$.

Exercise 11

Solve $y' - 2y = f(t)$ with $y(0) = 0$, where $f(t) = \begin{cases} 2 & t < 1 \\ -2 & t \geq 1. \end{cases}$

Exercise 12 (Laguerre's Equation)

Solve $ty'' - (1-t)y' + ny = 0$ by applying the Laplace transform and then solving the linear equation with respect to $Y(s)$.

Exercise 13

Solve $f(t) + \displaystyle\int_0^t f(\tau)\,d\tau = 1$.

Exercise 14

Solve $y(t) - 2\displaystyle\int_0^t y(t-\tau)e^{2t}\,d\tau = \cos 2t$.

Exercise 15 (Transform of Periodic Functions)

Prove that if $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt.$$

Exercise 16

Solve $y'' + 9y = \delta(t-\pi) + 2\delta(t-2\pi)$, $y(0) = 0$, $y'(0) = 0$.

Exercise 17

Solve the following system of differential equations

$$\begin{aligned} x'' + 9x - 2y &= 0 \\ y'' - 4x + 4y &= 0 \end{aligned}$$

where $x(0) = 0$, $x'(0) = 1$, $y(0) = 0$, $y'(0) = -2$.