# Dirichlet's Theorem on Arithmetic Progressions

MATH 300 HNR Foundations of Mathematics, Texas A&M University

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## 1 Contents at a Glance

Our goal is to prove the following theorem.

**Theorem 1.1**

If $\gcd(a, n) = 1$, then there are infinitely many primes of the form on the arithmetic progression $a + kn$.

The proof is done by proving that $\sum_{\substack{p \equiv a \pmod{d} \\ p \text{ prime}}} \frac{1}{p}$ diverges. It is sufficient to prove the following since if there are finitely many primes of the form $a + nd$, then $\sum_{\substack{p \equiv a \pmod{d} \\ p \text{ prime}}} \frac{1}{p}$ should converge.

To prove the following, we introduce a multiplicative function $\chi : U_n \to \mathbb{C}$. This is called a *Dirichlet character*.

**Definition 1.1: Dirichlet Character**

A **Dirichlet chatacter mod m** is a function, $\chi : U_m \to \mathbb{C}$ that is not identically zero and satisfies $\chi(ab) = \chi(a)\chi(b)$.

This definition can be extended to all of $\mathbb{Z}$ by letting $\chi(a) = 0$ if $\gcd(a, m) > 1$ and $\chi(a + m) = \chi(a)$). With Dirichlet characters, Riemann-Zeta functions, and Dirichlet $L$-functions, we will prove that $\sum_{\substack{p \equiv a \pmod{d} \\ p \text{ prime}}} \frac{1}{p}$ diverges.

## 2 The Riemann-Zeta Function

**Definition 2.1: Riemann-Zeta Function**

The **Riemann-Zeta function** $\zeta : \mathbb{R} \to \mathbb{R}_{>0}$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

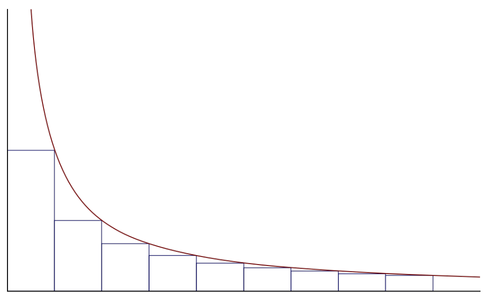
We now want to look at for which values of $s$ the function is defined. That is, for which values of $s$ the series converge. This can be found by the integral test.

**Lemma : Integral Test**

Suppose that $f(x)$ is a continuous, positive, and decreasing function on the interval $[k, \infty)$ for some $k \in \mathbb{Z}$ and let $f(n) = a_n$. Then

$$\int_k^{\infty} f(x)\, dx \text{ converges if and only if } \sum_{n=k}^{\infty} a_n \text{ converges.}$$

*Proof.* Note that by shifting, it is sufficient to prove the case when $k = 1$.

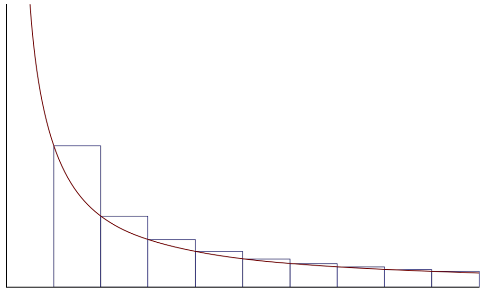


Let $f(x)$ be the red curve. The integral $\displaystyle\int_1^{\infty} f(x)\, dx$ is equal to the area between $f(x)$ and the $x$-axis on the interval $[1, \infty)$, which is clearly greater than

$$\sum_{n=2}^{\infty} 1 \cdot a_n = 1 \cdot a_2 + 1 \cdot a_3 + 1 \cdot a_4 + \cdots.$$

Furthermore, the area is less than

$$\sum_{n=1}^{\infty} 1 \cdot a_n = 1 \cdot a_1 + 1 \cdot a_2 + 1 \cdot a_3 + \cdots.$$



If $\int_1^{\infty} f(x)\ dx$ diverges, then

$$\sum_{n=1}^{\infty} a_n > \int_1^{\infty} f(x)\ dx$$

will also diverge, and if $\int_1^{\infty} f(x)\ dx$ converge, then

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} < a_1 + \int_1^{\infty} f(x)\ dx$$

will also converge since $a_1$ is finite. □

**Theorem 2.1**

$\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$. Thus, the domain of $\zeta(s)$ is $s > 1$.

*Proof.* It is sufficient to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if $s > 1$.

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,
\end{aligned}$$

which goes to infinity. For $s > 1$, if we let $f(x) = x^{-s}$,

$$\int_1^\infty x^{-s}\, dx = \frac{1}{-s+1} x^{-s+1} \Big|_1^\infty = \frac{1}{s-1}.$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s > 1$ by the integral test. ☐

**Lemma : Basel Problem**

$\zeta(2) = \frac{\pi^2}{6}$.

*Proof.* We compute $\zeta(2)$ by approximating $\sin x$ as polynomials. The Taylor series of $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

We have another way to find a polynomial expression of $\sin x$. Since the roots of $\sin x$ are $\cdots$, $-2\pi$, $-\pi$, $0$, $\pi$, $2\pi$, $\cdots$, we can express $\sin x$ as

$$\sin x = ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots.$$

From the identity $\lim_{x\to 0} \frac{\sin x}{x} = 1$ , we can find $a$, and

$$\begin{aligned}\sin x &= x\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{2\pi}\right)\cdots \\ &= x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots.\end{aligned}$$

Therefore, we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots.$$

Comparing the coefficients of $x^3$, we get

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} = -\frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right).$$

So

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

☐

**Lemma**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

*Proof.* Recall the sum of geometric series formula

$$1 + r + r^2 + \cdots = \frac{1}{1-r}$$

if $|r| < 1$. Since $\left|\frac{1}{p^s}\right| < 1$, we have

$$\begin{aligned}
\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \cdots\right)\cdots \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s}.
\end{aligned}$$

where the second last line follows from the unique prime factorization of positive integers greater than 1. □

**Theorem 2.2**

$$\sum_{p \text{ prime}} \frac{1}{p} \text{ diverges.}$$

Before the proof, we note that if $z$ is some complex number such that $|z| < 1$, then the Taylor series for $-\log(1-z)$ is

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

where log denotes the natural logarithm.

*Proof.* Note that $\left|\frac{1}{p^s}\right| < 1$. We have

$$\log \zeta(s) = \log \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= -\sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^s}\right)$$

$$= \sum_{p \text{ prime}} \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \cdots\right)$$

$$= \sum_{p \text{ prime}} \frac{1}{p^s} + \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^{ns}}$$

**Claim.** $\sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n}$ is bounded above by $\frac{\pi^2}{6}$.

Since

$$\sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n} = \sum_{p \text{ prime}} \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots$$

$$\leq \sum_{p \text{ prime}} \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots$$

$$= \sum_{p \text{ prime}} \frac{1}{p^2}\left(\frac{1}{1 - \frac{1}{p}}\right)$$

$$\leq \sum_{p \text{ prime}} \frac{1}{(p-1)^2}$$

$$< \sum_{n \in \mathbb{N}} \frac{1}{n^2},$$

$\sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n}$ is bounded above by $\frac{\pi^2}{6}$. We also see that this is bounded below by 0 since all terms of the series is positive. Therefore,

$$\sum_{p \text{ prime}} \frac{1}{p} = \lim_{s \to 1^+} \sum_{p \text{ prime}} \frac{1}{p^s}$$

$$= \lim_{s \to 1^+} \log \zeta(s) - \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n}$$

$$> \lim_{s \to 1^+} \log \zeta(s) - \frac{\pi^2}{6},$$

which diverges because $\log \zeta(1)$ diverges. $\square$

## 3 Dirichlet Characters

**Definition 3.1: Dirichlet Character**

A **Dirichlet chatacter mod m** is a function, $\chi : U_m \to \mathbb{C}$ that is not identically zero and satisfies $\chi(ab) = \chi(a)\chi(b)$.

We extend $\chi$ to all of $\mathbb{Z}$ by letting $\chi(a) = 0$ if $\gcd(a, m) > 1$ and periodicity, i.e. $\chi(a + m) = \chi(a)$).

**Corollary**

Dirichlet characters are totally multiplicative. That is, $\chi(ab) = \chi(a)\chi(b)$ for all $a$, $b \in \mathbb{Z}$.

For every Dirichlet character, $\chi(1)$ should be 1. This is because

$$\chi(1)^2 = \chi(1^2) = \chi(1),$$

so $\chi(1) = 0$ or 1. However, if $\chi(1) = 0$, then $\chi$ is identically zero. Thus $\chi(1) = 1$ for every Dirichlet character.

From now on, we use the notation $\chi_0$ for the *principal chatacter*, which is

$$\chi_0(a) = \begin{cases} 1 & \gcd(a, m) = 1 \\ 0 & \gcd(a, m) > 1 \end{cases}.$$

**Example 1**

There are exactly two Dirichlet characters mod 3:

$$\chi_0(a) = \begin{cases} 1 & a \equiv 1 \pmod 3 \\ 1 & a \equiv 2 \pmod 3 \\ 0 & a \equiv 0 \pmod 3 \end{cases} \quad \text{and} \quad \chi_1(a) = \begin{cases} 1 & a \equiv 1 \pmod 3 \\ -1 & a \equiv 2 \pmod 3 \\ 0 & a \equiv 0 \pmod 3 \end{cases}.$$

**Example 2**

There are exactly two Dirichlet characters mod 4:

$$\chi_0(a) = \begin{cases} 1 & a \equiv 1 \pmod 4 \\ 0 & a \equiv 2 \pmod 4 \\ 1 & a \equiv 3 \pmod 4 \\ 0 & a \equiv 0 \pmod 4 \end{cases} \quad \text{and} \quad \chi_1(a) = \begin{cases} 1 & a \equiv 1 \pmod 4 \\ 0 & a \equiv 2 \pmod 4 \\ -1 & a \equiv 3 \pmod 4 \\ 0 & a \equiv 0 \pmod 4 \end{cases}.$$

**Example 3**

There are exactly four Dirichlet characters mod 5:

$$\chi_0(a) = \begin{cases} 1 & a \equiv 1 \pmod 5 \\ 1 & a \equiv 2 \pmod 5 \\ 1 & a \equiv 3 \pmod 5 \\ 1 & a \equiv 4 \pmod 5 \\ 0 & a \equiv 0 \pmod 5 \end{cases}, \qquad \chi_1(a) = \begin{cases} 1 & a \equiv 1 \pmod 5 \\ i & a \equiv 2 \pmod 5 \\ -i & a \equiv 3 \pmod 5 \\ -1 & a \equiv 4 \pmod 5 \\ 0 & a \equiv 0 \pmod 5 \end{cases}$$

$$\chi_2(a) = \begin{cases} 1 & a \equiv 1 \pmod 5 \\ -1 & a \equiv 2 \pmod 5 \\ -1 & a \equiv 3 \pmod 5 \\ 1 & a \equiv 4 \pmod 5 \\ 0 & a \equiv 0 \pmod 5 \end{cases}, \qquad \chi_3(a) = \begin{cases} 1 & a \equiv 1 \pmod 5 \\ -i & a \equiv 2 \pmod 5 \\ i & a \equiv 3 \pmod 5 \\ -1 & a \equiv 4 \pmod 5 \\ 0 & a \equiv 0 \pmod 5 \end{cases}.$$

**Definition 3.2: Dirichlet $L$-function**

The **Dirichlet $L$-function** is defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

**Lemma**

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

*Proof.* We have

$$\begin{aligned}
\prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} &= \prod_{p \text{ prime}} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}} + \cdots\right) \\
&= \prod_{p \text{ prime}} \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots\right) \\
&= \left(1 + \frac{\chi(2)}{2^s} + \frac{\chi(2^2)}{2^{2s}} + \cdots\right)\left(1 + \frac{\chi(3)}{3^s} + \frac{\chi(3^2)}{3^{2s}} + \cdots\right)\cdots \\
&= 1 + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(2^2)}{4^s} + \frac{\chi(5)}{5^s} + \frac{\chi(2)\chi(3)}{6^s} + \cdots \\
&= 1 + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \frac{\chi(5)}{5^s} + \frac{\chi(6)}{6^s} + \cdots
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

where the third last line follows from the unique prime factorization of positive integers greater than 1. □

**Theorem 3.1**

$$\sum_{\substack{p \equiv 1 \pmod 4 \\ p \text{ prime}}} \frac{1}{p} \text{ diverges.}$$

*Proof.* We consider the natural log of $L(1, \chi)$.

$$\begin{aligned}
\log L(1, \chi) &= \log \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \\
&= \sum_{p \text{ prime}} -\log \left(1 - \frac{\chi(p)}{p}\right) \\
&= \sum_{p \text{ prime}} \left(\frac{\chi(p)}{p} + \frac{\chi(p^2)}{2p^2} + \frac{\chi(p^3)}{3p^3} + \cdots\right) \\
&= \sum_{p \text{ prime}} \frac{\chi(p)}{p} + \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{\chi(p^n)}{np^n}
\end{aligned}$$

We see that

$$\left| \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{\chi(p^n)}{np^n} \right| \leq \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{|\chi(p^n)|}{np^n} \leq \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n}$$

by the triangle inequality, so $\sum_{p \text{ prime}} \sum_{n \geq 2} \frac{\chi(p^n)}{np^n}$ is bounded by $-\frac{\pi^2}{6}$ and $\frac{\pi^2}{6}$. Then

$$\begin{aligned}
\lim_{s \to 1^+} \log \zeta(s) + \log L(1, \chi) &= \sum_{p \text{ prime}} \frac{1}{p} + \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{1}{np^n} \\
&\quad + \sum_{p \text{ prime}} \frac{\chi(p)}{p} + \sum_{p \text{ prime}} \sum_{n \geq 2} \frac{\chi(p^n)}{np^n} \\
&= 2 \sum_{\substack{p \equiv 1 \pmod 4 \\ p \text{ prime}}} \frac{1}{p} + c
\end{aligned}$$

where $c$ is a bounded constant. Since the left-hand side diverges, the right side should also diverge, which implies that $\displaystyle\sum_{\substack{p\equiv 1 \pmod 4\\ p \text{ prime}}} \frac{1}{p}$ diverges. □

This is one of the proofs that there are infinitely many primes that are of the form $1+4n$. Similar argument can be used for primes of the form $3+4n$.

**Theorem 3.2**

$\displaystyle\sum_{\substack{p\equiv 3 \pmod 4\\ p \text{ prime}}} \frac{1}{p}$ diverges.

*Proof.*

**Claim.** $\lim_{s\to 1^+} \log\zeta(s) - \log L(1,\chi)$ diverges.

It is sufficient to show that $\displaystyle\lim_{s\to 1^+} \frac{\zeta(s)}{L(s,\chi)}$ diverges. We have

$$\begin{aligned} L(s,\chi) &= \sum_{n\equiv 1 \pmod 4} \frac{1}{n^s} - \sum_{n\equiv 3 \pmod 4} \frac{1}{n^s} \\ &= \zeta(s) - \sum_{2|n} \frac{1}{n^s} - 2\sum_{n\equiv 3 \pmod 4} \frac{1}{n^s} \\ &\geq \zeta(s) - \sum_{2|n} \frac{1}{n^s} - 2\sum_{4|n} \frac{1}{n^s} \\ &= \left(1 - \frac{1}{2^s} - \frac{2}{4^s}\right)\zeta(s), \end{aligned}$$

so

$$\lim_{s\to 1^+} \frac{\zeta(s)}{L(s,\chi)} \geq \lim_{s\to 1^+} \frac{4^s}{4^s - 2^s - 2},$$

and this diverges. We have

$$\log\zeta(s) - \log L(s,\chi) = 2\sum_{\substack{p\equiv 3 \pmod 4\\ p \text{ prime}}} \frac{1}{p^s} + c,$$

where $c$ is a bounded constant. Letting $s \to 1^+$, since the left hand side diverges, $\displaystyle\sum_{\substack{p\equiv 3 \pmod 4\\ p \text{ prime}}} \frac{1}{p}$ also diverges. □

## 4 Generalizations

Denote $\mathbb{D}_m$ as the set of all Dirichlet characters modulo $m$. Note that $U_m$ forms a group under multiplication, and $\mathbb{D}_m$ also forms a group under the binary operation $(\chi_1 * \chi_2)(a) = \chi_1(a)\chi_2(a)$.

**Theorem 4.1**

There are exactly $\varphi(m)$ Dirichlet characters modulo $m$.

**Solution** Note that $U_m$ has $\varphi(m)$ elements.

> **Claim.** $U_m$ is isomorphic to $\mathbb{D}_m$.

Let $m = 2^t p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$. Then, $U_m = U_{2^t} \times U_{p_1^{k_1}} \times \cdots \times U_{p_s^{k_s}}$.

- if $t = 1$, then $U_2$ is generated by 1.
- if $t = 2$, then $U_4$ is generated by 3.
- if $t > 2$, then since 3 has order $2^{t-2}$, and -1 is not a power of 3, $U_{2^t}$ is generated by 3 and -1.

Also, we can find generators of each $U_{p_i^{k_i}}$ because $U_{p_1^{k_1}}$, $U_{p_2^{k_2}}$, …, $U_{p_s^{k_s}}$ are cyclic. Finally, the mapping

$$\begin{aligned} 3^a(-1)^b g_1^{l_1} g_2^{l_2} \cdots g_s^{l_s} &\mapsto \left( e^{\frac{2\pi i c}{\varphi(2^t)}}, e^{\frac{2\pi i l_1}{\varphi(p_1^{k_1})}}, e^{\frac{2\pi i l_2}{\varphi(p_2^{k_2})}}, \ldots, e^{\frac{2\pi i l_s}{\varphi(p_s^{k_s})}} \right) \\ &\mapsto (\chi(c), \chi(g_1), \chi(g_2), \ldots, \chi(g_s)) \\ &\mapsto \chi \end{aligned}$$

implies that $U_m$ is isomorphic to $\mathbb{D}_m$.

Thus, $U_m$ and $\mathbb{D}_m$ has the same cardinality, which is $\varphi(m)$.

**Lemma**

If $\chi$ is a Dirichlet character mod $m$, then $\overline{\chi}$, the function that takes conjugate values of $\chi$, is also a Dirichlet character.

*Proof.* We have

$$\overline{\chi}(ab) = \overline{\chi(ab)} = \overline{\chi(a)\chi(b)} = \overline{\chi(a)} \cdot \overline{\chi(b)} = \overline{\chi}(a)\overline{\chi}(b).$$

Also, if $\gcd(a, m) > 1$, then $\overline{\chi(a)} = \overline{0} = 0$, and since

$$\overline{\chi}(a + m) = \overline{\chi(a + m)} = \overline{\chi(a)} = \overline{\chi}(a),$$

$\overline{\chi}$ is also a Dirichlet character. □

**Theorem 4.2**

$L(1, \chi_0)$ diverges.

*Proof.* We see that

$$\begin{aligned} L(s, \chi_0) &= \prod_{p \text{ prime}} \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} \\ &= \prod_{p \nmid m} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \mid m} \left(1 - \frac{0}{p^s}\right)^{-1} \\ &= \prod_{p \nmid m} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_{p \nmid m} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \frac{\prod_{p \mid m} \left(1 - \frac{1}{p^s}\right)^{-1}}{\prod_{p \mid m} \left(1 - \frac{1}{p^s}\right)^{-1}} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \prod_{p \mid m} \left(1 - \frac{1}{p^s}\right) \\ &= \zeta(s) \cdot \prod_{p \mid m} \left(1 - \frac{1}{p^s}\right). \end{aligned}$$

Here, $\prod_{p \mid m} \left(1 - \frac{1}{p^s}\right)$ should be finite since there are finite prime divisors of $m$, thus it is a finite product of finite terms. Therefore

$$\lim_{s \to 1^+} L(s, \chi_0) = \infty$$

since $\zeta(s)$ diverges as $s \to 1^+$. □

**Lemma**

$$\sum_{\chi \in \mathbb{D}_m} \chi(n) = \begin{cases} \varphi(m) & n \equiv 1 \mod m \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Take any character $\psi$. Then $\chi(n) = \psi(n)\psi^{-1}(n)\chi(n)$.

**Claim.** $\psi^{-1}(n)\mathbb{D}_m = \{\psi^{-1}(n)\chi(n) \mid \chi \in \mathbb{D}_m\} = \mathbb{D}_m$.

Note that the set $\{\psi^{-1}(n)\chi(n) \mid \chi \in \mathbb{D}_m\}$ also forms a group under the binary operation of $\mathbb{D}_m$, with $\psi$ the identity. If $\psi^{-1} * \chi_1 = \psi^{-1} * \chi_2$, then $\chi_1 = \chi_2$ by the left cancellation property. This gives that the set $\{\psi^{-1}(n)\chi(n) \mid \chi \in \mathbb{D}_m\}$ is equal to $\mathbb{D}_m$.

We have

$$\sum_{\chi \in \mathbb{D}_m} \chi(n) = \psi(n) \sum_{\chi \in \mathbb{D}_m} \psi^{-1}(n)\chi(n) = \psi(n) \sum_{\chi \in \mathbb{D}_m} \chi(n),$$

which implies that either $\psi(n) \neq 1$ and $\sum_{\chi \in \mathbb{D}_m} \chi(n) = 0$ , or $\psi(n) = 1$ for all $\psi \in \mathbb{D}_m$.

If $n \equiv 1 \pmod m$, then $\chi(n) = 1$ for all $\chi \in \mathbb{D}_m$. Thus

$$\sum_{\chi \in \mathbb{D}_m} \chi(n) = \sum_{\chi \in \mathbb{D}_m} 1 = \varphi(m).$$

**Claim.** If $n \not\equiv 1 \pmod m$, then there is some $\chi \in \mathbb{D}_m$ such that $\chi(n) \neq 1$.

If $n \not\equiv 1 \mod m$, then there exists $l_1, l_2, \ldots, l_r \neq 0$ such that $n = g_1^{l_1} g_2^{l_2} \cdots g_r^{l_r}$. Fix $s \in \{1, 2, \ldots, r\}$, and take the character that has

$$\chi(g_j) = \begin{cases} 1 & j \neq s \\ e^{\frac{2\pi i}{\varphi(p_s^{k_s})}} & j = s \end{cases}$$

This gives $\chi(n) \neq 1$, and hence $\sum_{\chi \in \mathbb{D}_m} \chi(n) = 0$, as desired. □

**Theorem 4.3**

$$\sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}n) = \begin{cases} \varphi(m) & n \equiv a \mod m \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If $n \equiv a \pmod m$, then $a^{-1}n \equiv 1 \pmod m$, so $\sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}n) = \varphi(m)$. If $n \not\equiv a \pmod m$, then $a^{-1}n \not\equiv 1 \pmod m$, so $\sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}n) = 0$. □

**Theorem 4.4**

If $\chi$ is any Dirichlet character mod $m$, then $L(1, \chi) \neq 0$.

*Proof.* Recall that if $\chi$ is a Dirichlet character, then $\overline{\chi}$ is also a Dirichlet character. So $L(s,\overline{\chi}) = \overline{L(s,\chi)} = \overline{L(s,\chi)}$, which shows that the values of $L(s,\chi)$ come in conjugate pairs and hence $\prod_{\chi\in\mathbb{D}_m} L(s,\chi)$ is real. Since $\log L(s,\chi) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^{ns}}$, we have

$$\begin{aligned}\frac{1}{\varphi(m)} \sum_{\chi\in\mathbb{D}_m} \log L(s,\chi) &= \frac{1}{\varphi(m)} \sum_{\chi\in\mathbb{D}_m} \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^{ns}} \\ &= \sum_{\substack{p^n\equiv 1 \pmod{m} \\ p \text{ prime}}} \sum_{n=1} \frac{1}{np^{ns}}\end{aligned}$$

If we set $s$ real and $s > 1$, then since the right hand side is real and nonnegative, the left hand side should also be real nonnegative. This gives

$$\sum_{\chi\in\mathbb{D}_m} \log L(s,\chi) \geq 0$$

and thus

$$\prod_{\chi\in\mathbb{D}_m} L(s,\chi) \geq 1.$$

Therefore $L(s,\chi) \neq 0$, and setting $s \to 1^+$ gives $L(1,\chi) \neq 0$. □

Note that the proof is actually not complete: to finish the proof with stating that $L(s,\chi) \neq 0$ as $s \to 1^+$, we need to use the fact that the function $L(s,\chi)$ has a meromorphic continuation to $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$, with one simple pole at $s = 1$. This part is omitted since it is out of our boundary to prove this fact.

**Theorem 4.5**

$\sum_{\substack{p\equiv a \pmod{m} \\ p \text{ prime}}} \frac{1}{p}$ diverges.

*Proof.* We have

$$\begin{aligned}\sum_{\substack{p\equiv a \pmod{m} \\ p \text{ prime}}} \frac{1}{p} &= \frac{1}{\varphi(m)} \sum_{p \text{ prime}} \sum_{x\in\mathbb{D}_m} \frac{\chi(a^{-1}p)}{p} \\ &= \frac{1}{\varphi(m)} \sum_{x\in\mathbb{D}_m} \sum_{p \text{ prime}} \frac{\chi(a^{-1}p)}{p} \\ &= \frac{1}{\varphi(m)} \sum_{x\in\mathbb{D}_m} \chi(a^{-1}) \sum_{p \text{ prime}} \frac{\chi(p)}{p}.\end{aligned}$$

Recall that $\log L(s,\chi) = \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} + \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p^n)}{np^{ns}}$, where $\sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p^n)}{np^{ns}}$ is bounded by $-\frac{\pi^2}{6}$ and $\frac{\pi^2}{6}$. This tells us that

$$\begin{aligned}
&\frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}) \log L(1,\chi) \\
&= \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}) \Bigg( \sum_{p \text{ prime}} \frac{\chi(p)}{p} + \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p^n)}{np^n} \Bigg) \\
&= \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \sum_{p \text{ prime}} \frac{\chi(a^{-1}p)}{p} + \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}) \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p^n)}{np^n} \\
&= \sum_{\substack{p \equiv a \pmod m \\ p \text{ prime}}} \frac{1}{p} + \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}) \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p^n)}{np^n} \\
&= \sum_{\substack{p \equiv a \pmod m \\ p \text{ prime}}} \frac{1}{p} + c
\end{aligned}$$

where $c$ is bounded since it is a finite sum of bounded terms. Thus

$$\begin{aligned}
\sum_{\substack{p \equiv a \pmod m \\ p \text{ prime}}} \frac{1}{p} &= \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m} \chi(a^{-1}) \log L(1,\chi) - c \\
&= \frac{1}{\varphi(m)} \log L(1,\chi_0) + \frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m \setminus \{\chi_0\}} \chi(a^{-1}) \log L(1,\chi) - c,
\end{aligned}$$

and $\sum_{\substack{p \equiv a \pmod m \\ p \text{ prime}}} \frac{1}{p}$ would diverge to infinity unless $\frac{1}{\varphi(m)} \sum_{\chi \in \mathbb{D}_m \setminus \{\chi_0\}} \chi(a^{-1}) \log L(1,\chi)$ is real and diverges to negative infinity.

> **Claim.** $\sum_{\chi \in \mathbb{D}_m \setminus \{\chi_0\}} \chi(a^{-1}) \log L(1,\chi)$ is real and does not diverge to negative infinity.

Let $\mathbb{D}_{m,\mathbb{R}}$ be the set of nontrivial Dirichlet characters mod $m$ that has only real values, and $\mathbb{D}_{m,\mathbb{C}}$ the set of Dirichlet characters mod $m$ that has complex values. Then

$$\sum_{\chi \in \mathbb{D}_m \setminus \{\chi_0\}} \chi(a^{-1}) \log L(1,\chi) = \sum_{\chi \in \mathbb{D}_{m,\mathbb{R}}} \chi(a^{-1}) \log L(1,\chi) + \sum_{\chi \in \mathbb{D}_{m,\mathbb{C}}} \chi(a^{-1}) \log L(1,\chi).$$

Here, $\displaystyle\sum_{\chi\in\mathbb{D}_{m,\mathbb{R}}} \chi(a^{-1})\log L(1,\chi)$ does not diverge to negative infinity since $L(1,\chi)\neq 0$ for all $\chi\in\mathbb{D}_{m,\mathbb{R}}$. For complex values, recall that if $\chi$ is a Dirichlet character, then $\overline{\chi}$ is also a Dirichlet character. So

$$\overline{\chi}(a^{-1})\log L(1,\overline{\chi}) = \overline{\chi(a^{-1})}\cdot\overline{\log L(1,\chi)} = \overline{\chi(a^{-1})\log L(1,\chi)}.$$

This tells that the values of $\chi(a^{-1})\log L(1,\chi)$ come in conjugate pairs, and hence $\displaystyle\sum_{\chi\in\mathbb{D}_{m,\mathbb{C}}} \chi(a^{-1})\log L(1,\chi)$ is real. Since $L(1,\chi)\neq 0$, $\displaystyle\sum_{\chi\in\mathbb{D}_{m,\mathbb{C}}} \chi(a^{-1})\log L(1,\chi)$ does not diverge to negative infinity. Thus

$$\sum_{\chi\in\mathbb{D}_m\setminus\{\chi_0\}} \chi(a^{-1})\log L(1,\chi) = \sum_{\chi\in\mathbb{D}_{m,\mathbb{R}}} \chi(a^{-1})\log L(1,\chi) + \sum_{\chi\in\mathbb{D}_{m,\mathbb{C}}} \chi(a^{-1})\log L(1,\chi)$$

is real which does not diverge to negative infinity. Therefore, in the formula

$$\sum_{\substack{p\equiv a \pmod m\\ p\text{ prime}}} \frac{1}{p} = \frac{1}{\varphi(m)}\chi_0(a^{-1})\log L(1,\chi_0) + \frac{1}{\varphi(m)}\sum_{\chi\in\mathbb{D}_m\setminus\{\chi_0\}} \chi(a^{-1})\log L(1,\chi) - c,$$

the right hand side is real and diverges to positive infinity, so the left hand side should also diverge to positive infinity, which completes the proof. □

This gives us the Dirichlet's theorem, as desired.

**Corollary : Dirichlet's Theorem on Arithmetic Progressions**

If $\gcd(a,m)=1$, then there are infinitely many primes in the arithmetic progression $a+km$.

## 5 Citations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