Several Variable Calculus

MATH 221 HNR, Texas A&M University

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Contents

1 Vector Geometry and 3-dimensional Space

1.1 Vectors

Definition 1.1: Vector (Geometric Definition)

An n-dimensional **vector** $\overrightarrow{v} \in \mathbb{R}^n$ is an arrow of specific length and direction.

Definition 1.2: Vector (Algebraic Definition)

An n-dimensional **vector** $\overrightarrow{v} \in \mathbb{R}^n$ is an ordered n-tuple of numbers

$$\overrightarrow{v} = \langle v_1, v_2, \cdots, v_n \rangle.$$

The v_is are called the **components** of $\overrightarrow{v}$.

A single point $A \in \mathbb{R}^n$ always corresponds to a vector $\overrightarrow{A} = \overrightarrow{OA}$, where O is the origin. Here $\overrightarrow{A}$ is called the *position vector* of A.

Note that the placement of $\overrightarrow{v}$ does not matter. Two vectors are considered the same if they have the same length and direction, and

Definition 1.3: Equal Vectors

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be vectors. Then the following are equivalent.

1. $\overrightarrow{v} = \overrightarrow{w}$
2. $\overrightarrow{w}$ is some translate of $\overrightarrow{v}$
3. The components of $\overrightarrow{v}$ and $\overrightarrow{w}$ match.

Definition 1.4: Zero Vector

The **zero vector** $\overrightarrow{0} \in \mathbb{R}^n$ is a vector with no length.

Definition 1.5: Magnitude

The **magnitude** of $\overrightarrow{v} \in \mathbb{R}^n$, denoted $|\overrightarrow{v}|$, is the length of $\overrightarrow{v}$. Algebraically,

$$|\overrightarrow{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Basic Vector Arithmetic / Algebra

There are two basic vector arithmetic.

Definition 1.6: Scalar Multiplication

Given $\overrightarrow{v} \in \mathbb{R}^n$ and scalar $\lambda \in \mathbb{R}$, $\lambda\overrightarrow{v}$ is a new vector defined by

$$\lambda\langle v_1, v_2, \cdots, v_n\rangle = \langle \lambda v_1, \lambda v_2, \cdots, \lambda v_n\rangle$$

or scaling the length of $\overrightarrow{v}$ by λ.

If $\lambda < 0$, then the new vector is in different dimension.

Definition 1.7: Vector Addition

Given $\overrightarrow{v}$ and $\overrightarrow{w} \in \mathbb{R}^n$, $\overrightarrow{v} + \overrightarrow{w}$ is the new vector defined by

$$\overrightarrow{v} + \overrightarrow{w} = \langle v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n\rangle$$

or attach $\overrightarrow{w}$ to the end of $\overrightarrow{v}$, then $\overrightarrow{v} + \overrightarrow{w}$ starts at $\overrightarrow{v}$ and ends at the end of $\overrightarrow{w}$.

For vector subtraction, $\overrightarrow{v} - \overrightarrow{w}$ can be interpreted as $\overrightarrow{v} + (-1)\overrightarrow{w}$.

Theorem 1.1

In any $\mathbb{R}^n$, the following holds:

- Commutativity: $\overrightarrow{v} + \overrightarrow{w} = \overrightarrow{w} + \overrightarrow{v}$
- Associativity: $\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w}$, $\lambda(\mu\overrightarrow{v}) = (\lambda\mu)\overrightarrow{v}$
- Distribution: $\lambda(\overrightarrow{v} + \overrightarrow{w}) = \lambda\overrightarrow{v} + \lambda\overrightarrow{w}$.

Theorem 1.2

The following holds for vectors.

1. $|\lambda\overrightarrow{v}| = |\lambda| \cdot |\overrightarrow{v}|$
2. $|\overrightarrow{v} + \overrightarrow{w}| \leq |\overrightarrow{v}| + |\overrightarrow{w}|$.

The second inequality is called the *triangle inequality*.

Proof.

$$\begin{aligned}
|\lambda\overrightarrow{v}| &= \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2 + \cdots + \lambda^2 v_n^2} \\
&= \sqrt{\lambda^2(v_1^2 + v_2^2 + \cdots + v_n^2)} \\
&= \sqrt{\lambda^2}\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\
&= |\lambda| \cdot |\overrightarrow{v}|.
\end{aligned}$$

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Definition 1.8: Parallel Vectors

Two vectors $\vec{v}$ and $\vec{w}$ are **parallel**, denoted $\vec{v} \parallel \vec{w}$, means that $\vec{v} = \lambda\vec{w}$ for some $\lambda \in \mathbb{R}$, or $\vec{v}$ and $\vec{w}$ point in same or opposite directions.

Definition 1.9: Unit Vector

A **unit vector** is a vector $\vec{e} \in \mathbb{R}^n$ with $|\vec{e}| = 1$. In $\mathbb{R}^2$, this implies $\vec{e} = \langle\cos\theta, \sin\theta\rangle$ for some θ.

Note that, given any $\vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$, we can define

$$\vec{e}_{\vec{v}} = \frac{1}{|\vec{v}|}\vec{v}$$

as a unit vector in the same direction as $\vec{v}$. Then we have $\vec{v} = |\vec{v}|\vec{e}_{\vec{v}}$.

We will use the notation

$$\vec{i} = \langle 1, 0\rangle,\ \vec{j} = \langle 0, 1\rangle \text{ in } \mathbb{R}^2$$

and

$$\vec{i} = \langle 1, 0, 0\rangle,\ \vec{j} = \langle 0, 1, 0\rangle,\ \vec{k} = \langle 0, 0, 1\rangle \text{ in } \mathbb{R}^3.$$

Then we have $\vec{v} = \langle v_1, v_2 = v_1\vec{i} + v_2\vec{j}$.

1.2 Dot Products

Definition 1.10: Dot Product (Algebraic)

Given $\vec{v} = \langle v_1, v_2, \cdots, v_n\rangle$ and $\vec{w} = \langle w_1, w_2, \cdots, w_n\rangle$ in $\mathbb{R}^n$, the **dot product** of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \cdots v_nw_n,$$

which is a real number.

Definition 1.11: Dot Product (Geometric)

Given $\vec{v}$ and $\vec{w}$, the **dot product** of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

Example 1

Find the angle between $\overrightarrow{a} = \langle 2, 2, -1 \rangle$ and $\overrightarrow{b} = \langle 5, -3, 2 \rangle$.

Solution We have $\overrightarrow{a} \cdot \overrightarrow{b} = 2$, so $\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$.

Remark.

In $n > 3$ dimensions, the angle between vectors is defied this way.

Remark.

Let θ be the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.

- θ is acute if and only if $\overrightarrow{v} \cdot \overrightarrow{w} > 0$
- θ is obtuse if and only if $\overrightarrow{v} \cdot \overrightarrow{w} < 0$.

Definition 1.12: Orthogonal

Two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are called **orthogonal**, denoted $\overrightarrow{v} \perp \overrightarrow{w}$ if $\overrightarrow{v} \cdot \overrightarrow{w} = 0$.

Then, what is the dot product really measuring?

Definition 1.13: Projection

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be two vectors in the same dimension. The **projection** of $\overrightarrow{v}$ onto $\overrightarrow{w}$ is defined by

$$\text{Proj}_{\overrightarrow{w}} \overrightarrow{v} = \left(\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{\overrightarrow{w} \cdot \overrightarrow{w}}\right) \overrightarrow{w}.$$

Here, the size of the projection is defined as the magnitude of the projection vector.

$$\text{Comp}_{\overrightarrow{w}} \overrightarrow{v} = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{\overrightarrow{w} \cdot \overrightarrow{w}} = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{|\overrightarrow{w}|}$$

Theorem 1.3: Algebraic Properties of the Dot Product

- $\overrightarrow{0} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{0} = 0$
- $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v}$ (Commutative)
- $(\lambda \overrightarrow{v}) \cdot \overrightarrow{w} = \lambda(\overrightarrow{v} \cdot \overrightarrow{w})$ (Associative)
- $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$ (Distributive)
- $\overrightarrow{v} \cdot \overrightarrow{v} = |\overrightarrow{v}|^2$

1.3 Cross Product

Definition 1.14: Cross Product (Algebraic)

The **cross product** (for $\mathbb{R}^3$ only) is a vector defined by the determinant

$$\begin{aligned}\overrightarrow{v} \times \overrightarrow{w} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2 w_3 - v_3 w_2)\overrightarrow{i} - (v_1 w_3 - v_3 w_1)\overrightarrow{j} + (v_1 w_2 - v_2 w_1)\overrightarrow{k}.\end{aligned}$$

Definition 1.15: Cross Product (Geometric)

Point fingers of right-hand towards $\overrightarrow{v}$, and palm towards $\overrightarrow{w}$. Then curl fingers ($\overrightarrow{v}$) towards palm ($\overrightarrow{w}$). The cross product vector $\overrightarrow{v} \times \overrightarrow{w}$ is a vector which has direction of the thumb, and length $|\overrightarrow{v} \times \overrightarrow{w}| = |\overrightarrow{v}||\overrightarrow{w}| \sin\theta$.

This is called the *right-hand rule.*

Remark.

Here, $|\overrightarrow{v} \times \overrightarrow{w}| = |\overrightarrow{v}||\overrightarrow{w}| \sin\theta$ is the area of the parallelogram defined by $\overrightarrow{v}$ and $\overrightarrow{w}$. So the area of $\triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Example 2

Find the area of $\triangle PQR$ where $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, and $R(1, -1, 1)$.

Solution We have $\overrightarrow{QP} = \langle 3, -1, 7\rangle$ and $\overrightarrow{QR} = \langle 3, -6, 2\rangle$. So

$$|\overrightarrow{QP} \times \overrightarrow{QR}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 7 \\ 3 & -6 & 2 \end{vmatrix} = 40\mathbf{i} + 15\mathbf{j} - 15\mathbf{k}.$$

Therefore the area is $\dfrac{1}{2}|\overrightarrow{QP} \times \overrightarrow{QR}| = \dfrac{5}{2}\sqrt{82}.$

Theorem 1.4: Algebraic Properties of the Cross Product

- $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$ (Anti-commutative)
- $(\lambda \vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda \vec{w})$ (Scalar Associative)
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$, and $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ (Distributive)
- $\vec{v} \times \vec{v} = \vec{0}$
- $\vec{v} \times \vec{w}$ if and only if $\vec{v} \parallel \vec{w}$ or one of $\vec{v}$ and $\vec{w}$ is $\vec{0}$.

Remark.
Cross product is not associative. So $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ in general.

Theorem 1.5

The volume of the parallelepiped formed by three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ is

$$\text{Volume} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

1.4 Basics of $\mathbb{R}^3$

- $z = 0$ is the xy-plane
- $z = a$ is the plane parallel to the xy-plane at $z = a$
- $y = 0$ is the zx-plane,

and we have other basic planes.

1.5 Cylinders

To draw a cylinder, when given an equation missing a variable, first assume that missing variable is zero, and draw the corresponding plane, and extend parallel to missing variable axis.

In general, one can use traces to understand shapes. Think then draw!

1.6 Lines in $\mathbb{R}^3$

Lines in $\mathbb{R}^n$ where $n > 2$ is easiest via vector. We have

$$\vec{r}(t) = \vec{p_0} + t \cdot \overrightarrow{\text{DIR}}$$

where $\overrightarrow{p_0}$ is a point on line and $\overrightarrow{\text{DIR}}$ is a vector parallel to. Since t is a parameter, different values of t gives different points on line.

Example 3

Find the vector parametrization for the line through $(2, 4, -3)$ and $(3, -7, 1)$. At what point does this intersect the xy-plane?

Solution We will use $\overrightarrow{p_0} = \langle 2, 4, -3\rangle$ and $\overrightarrow{\text{DIR}} = \langle 3, -1, 1\rangle - \langle 2, 4, -3\rangle = \langle 1, -5, 4\rangle$. So

$$\overrightarrow{r}(t) = \langle 2, 4, -3\rangle + t\langle 1, -5, 4\rangle.$$

When this line intersect with the xy-plane, $z = -3 + 4t$ should be zero, so $t = \dfrac{3}{4}$. The line intersects at

$$\overrightarrow{r}\left(\frac{3}{4}\right) = \left(\frac{11}{4}, \frac{1}{4}, 0\right).$$

Remark.

One line can have infinitely many different parametrizations.

Example 4

Do $\overrightarrow{r_1}(t) = \langle 1 + t, -2 + 3t, 4 - t\rangle$ and $\overrightarrow{r_2}(s) = \langle 2s, 3 + s, -3 + 4s\rangle$ intersect?

Solution Setting $\overrightarrow{r_1} = \overrightarrow{r_2}$ will give a system of equation with 3 equations and 2 variables. Since this system is not consistent, the lines do not intersect.

Remark.

If given more than one line, give each parameter a different name.

Note that two lines are parallel if and only if their direction vectors are parallel.

Definition 1.16: Skew

When two lines are not parallel but do not intersect, we call them **skew**.

Spheres

The scalar equation for a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

The vector equation for sphere centered at $\overrightarrow{c}$ and radius r is

$$|\overrightarrow{x} - \overrightarrow{c}| = r.$$

where $\overrightarrow{x} = \langle x, y, z\rangle$ is a variable vector tracking points on the sphere.

Planes

To construct a plane, we choose a starting point $\overrightarrow{p_0}$ on the plane. Then all directions from $\overrightarrow{p_0}$ are orthogonal to a single vector $\overrightarrow{n}$ which is orthogonal to the plane. So

$$(\overrightarrow{x} - \overrightarrow{p_0}) \cdot \overrightarrow{n} = 0$$

is the vector equation of the plane with point $\overrightarrow{p_0}$ and normal vector $\overrightarrow{n}$. If we write $\overrightarrow{n} = \langle a, b, c \rangle$, then

$$\begin{aligned} \overrightarrow{x} \cdot \overrightarrow{n} &= \overrightarrow{p_0} \cdot \overrightarrow{n} \\ ax + by + cd &= d. \end{aligned}$$

Definition 1.17: Parallel Planes

Two planes are **parallel** if and only if their normal vectors are parallel.

2 Vector Functions

2.1 Vector-Valued Functions

Definition 2.1: Vector-Valued Function

A **vector-valued function** (or 1 variable) is a function

$$\overrightarrow{r} : \text{(Domain in) } \mathbb{R} \to \mathbb{R}^n$$

written as

$$\overrightarrow{r}(t) = \langle x_1(t), x_2(t), \cdots, x_n(t) \rangle$$

For $n = 3$, we use the notation $\overrightarrow{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Example 1

$\overrightarrow{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ has domain $[0, 3)$.

The domain variable is also called a *parameter*, and one can view $\overrightarrow{r}(t)$ as n separate functions $x(t)$, $y(t)$, ... called *component functions*

How should we graph $\overrightarrow{r}(t)$? When drawing a graph for a 1-dimension funciton $y = f(x)$, we drew both the domain (x) and the codomain (y). For vector functions. We do not draw the domain. We draw outputs as a curve in the codomain $\mathbb{R}^n$, and we say that $\overrightarrow{r}(t)$ *parametrizes* the curve.

Example 2

Draw $\overrightarrow{r}(t) = \langle \cos t, \sin t, t \rangle$.

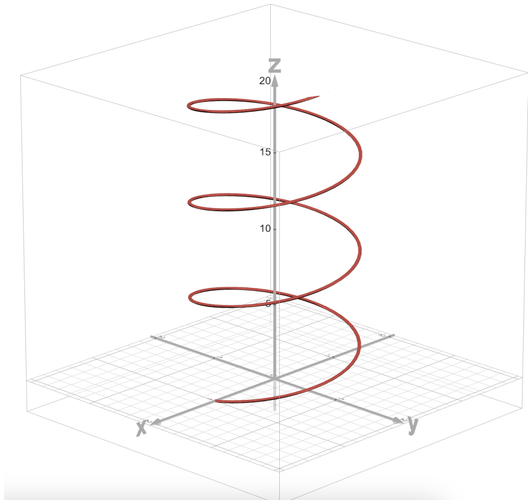
Solution The table for the vector value is:

t	$\overrightarrow{r}(t)$
0	$\langle 1, 0, 0 \rangle$
$\pi/2$	$\langle 0, 1, \pi/2 \rangle$
π	$\langle -1, 0, \pi \rangle$
$3\pi/2$	$\langle 0, -1, 3\pi/2 \rangle$
$-\pi/2$	$\langle 0, -1, -\pi/2 \rangle$

So drawing the graph gives the following diagram below.

Remark.

Multiple functions could parametrize the same curve in different ways.



> **Example 3**
> Find $\vec{r}(t)$ for the intersection of the cylinder $x^2 + y^2 = 1$ and plane $y + z = 2$.

Solution We try to solve for 1 variable in terms of 1 other variable. Sinze $z = 2 - y$ and $x^2 = 1 - y^2$, we let $y = t$ and this gives

$$\vec{r}(t) = \langle \pm\sqrt{1-t^2}, t, 2-t \rangle.$$

This is bad, and we can't do this because of the $\pm$ sign.

Solution We notice that $x^2 + y^2 = 1$ is satisfied by $x = \cos t$ and $y = \sin t$. Then $z = 2 - \sin t$, so

$$\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle.$$

2.2 Calculus of Vector-Valued Functions

Note that all of calculus are done component-wise.

Theorem 2.1

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle.$$

Definition 2.2: Continuous

$\vec{r}(t)$ is **continuous** at $t = t_0$ if $x(t)$, $y(t)$, and $z(t)$ are continuous at $t = t_0$.

Definition 2.3: Derivative

The **derivative** of a vector function is defined by

$$\frac{d}{dt}\overrightarrow{r}(t) = \overrightarrow{r}'(t) = \lim_{h\to 0}\frac{1}{h}\left(\overrightarrow{r}(t+h) - \overrightarrow{r}(t)\right).$$

Theorem 2.2

$\overrightarrow{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ and so on for $\overrightarrow{r}''(t)$ and higher derivatives.

Theorem 2.3

$\displaystyle\int \overrightarrow{r}(t)\ dt = \left\langle \int x(t)\ dt, \int y(t)\ dt, \int z(t)\ dt \right\rangle$, and similar for definite integrals.

Note that the integrals of the components has its own integration constants. So we can write this constant as vector $\langle c_1, c_2, c_3\rangle$, and write

$$\int \overrightarrow{r}(t)\ dt = \overrightarrow{R}(t) + \overrightarrow{c}.$$

Theorem 2.4: Fundamental Theorem of Calculus

- $\displaystyle\int_a^b \overrightarrow{r}'(t) = \overrightarrow{r}(b) - \overrightarrow{r}(a)$.
- $\displaystyle\frac{d}{dt}\left[\int_a^t \overrightarrow{r}(\tau)\ d\tau\right] = \overrightarrow{r}(t)$.

Theorem 2.5

- $\frac{d}{dt}\left[\overrightarrow{r_1}(t) \pm \overrightarrow{r_2}(t)\right] = \overrightarrow{r_1}'(t) \pm \overrightarrow{r_2}(t)$
- $\frac{d}{dt}\left[\lambda(t)\overrightarrow{r}(t)\right] = \lambda'(t)\overrightarrow{r}(t) + \lambda(t)\overrightarrow{r}'(t)$
- $\frac{d}{dt}\left[\overrightarrow{r_1}(t) \cdot \overrightarrow{r_2}(t)\right] = \overrightarrow{r_1}'(t) \cdot \overrightarrow{r_2}(t) + \overrightarrow{r_1}(t) \cdot \overrightarrow{r_2}'(t)$
- $\frac{d}{dt}\left[\overrightarrow{r_1}(t) \times \overrightarrow{r_2}(t)\right] = \overrightarrow{r_1}'(t) \times \overrightarrow{r_2}(t) + \overrightarrow{r_1}(t) \times \overrightarrow{r_2}'(t)$
- $\frac{d}{dt}\left[\overrightarrow{r}\big(g(t)\big)\right] = g'(t)\overrightarrow{r}'\big(g(t)\big).$

Example 4

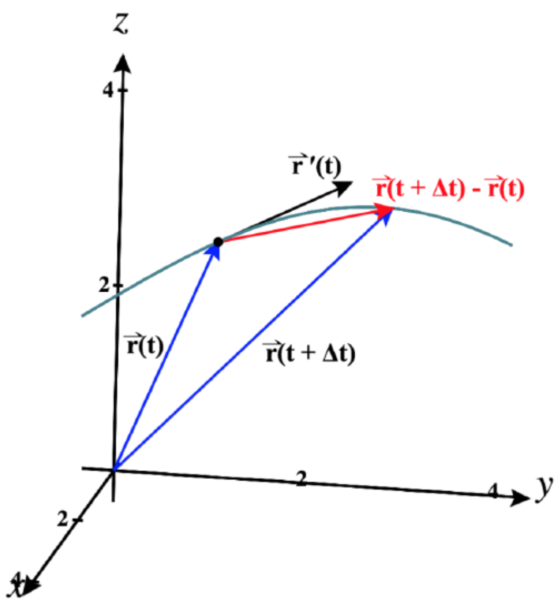
Show that if $|\overrightarrow{r}(t)| = c$ a constant, then $\overrightarrow{r} \perp \overrightarrow{r}'$.

Solution Recall that $|\overrightarrow{r}(t)|^2 = \overrightarrow{r} \cdot \overrightarrow{r}$. So $\overrightarrow{r} \cdot \overrightarrow{r} = c^2$. Taking the derivative of both sides, we get

$$\frac{d}{dt}\left[\overrightarrow{r} \cdot \overrightarrow{r}\right] = \overrightarrow{r}' \cdot \overrightarrow{r} + \overrightarrow{r} \cdot \overrightarrow{r}' = 2\overrightarrow{r} \cdot \overrightarrow{r}' = 0,$$

so $\overrightarrow{r} \cdot \overrightarrow{r}' = 0$. Therefore $\overrightarrow{r} \perp \overrightarrow{r}'$.

Then, what does the derivative say? We take a geometric approach.



If $\overrightarrow{r}(t_0)$ is a position vector P on the curve, then $\overrightarrow{r}'(t_0)$ is the tangent vector at P. Thus, if $P = \overrightarrow{r}(t_0)$ on curve, the tangent line at P can be parametrized by

$$\overrightarrow{L}(u) = \overrightarrow{r}(t_0) + u\overrightarrow{r}'(t_0).$$

Example 5

Let $\overrightarrow{r}(t) = \langle 2\cos t, \sin t, t\rangle$. Find the parametrization for the tangent line at $P = (0, 1, \pi/2)$.

Solution We have $\overrightarrow{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$. Since $t_0 = \pi/2$, we have $\overrightarrow{r}'(t_0) = \langle -2, 0, 1\rangle$ and

$$\overrightarrow{L}(u) = \langle 0, 1, \pi/2\rangle + u\langle -2, 0, 1\rangle.$$

2.3 Basic Differential Geometry

Definition 2.4: Arc Length

If $P = \overrightarrow{r}(a)$, $Q = \overrightarrow{r}(b)$ along curve $\mathcal{C}$, then the **arc length** along $\mathcal{C}$ from P to Q is

$$s = \left| \int_a^b \left|\overrightarrow{r}'(t)\right| \, dt \right|.$$

Definition 2.5: Signed Arc Length Function

Choose a specific starting point $P_0 = \overrightarrow{r}(t_0)$, then we define the **signed arc length funciton**

$$s(t) = \int_{t_0}^{t} \left|\overrightarrow{r}'(\tau)\right| \, d\tau.$$

Arc Length Parametrization

Definition 2.6: Smooth Curve

$\overrightarrow{r}(t)$ is **smooth** if $\overrightarrow{r}' \neq \overrightarrow{0}$.

Remark.

$\dfrac{ds(t)}{dt} = |\overrightarrow{r}'(t)|$.

Definition 2.7: Unit Tangent Vector

The **unit tangent vector** for curve $\mathcal{C}$ is

$$\overrightarrow{T}(t) = \frac{1}{|\overrightarrow{r}'(t)|}\overrightarrow{r}'(t).$$

or simply $\overrightarrow{T}(s) = \overrightarrow{r}'(s)$.

Definition 2.8: Curvature

The **curvature** of a curve $\mathcal{C}$ is

$$\kappa(s) = \left|\frac{d\overrightarrow{T}}{ds}\right|$$

or

$$\kappa(t) = \frac{\left|\overrightarrow{T}'(t)\right|}{|\overrightarrow{r}'(t)|}.$$

Theorem 2.6

$$\kappa(t) = \frac{|\overrightarrow{r}' \times \overrightarrow{r}''|}{|\overrightarrow{r}'|^3}.$$

Definition 2.9: Unit Normal Vector

The **unit normal vector** for $\mathcal{C}$ is

$$\overrightarrow{N} = \frac{1}{|\overrightarrow{T}'|}\overrightarrow{T}'.$$

Theorem 2.7

$\overrightarrow{N} \perp \overrightarrow{T}$.

This is because example 8 from section 2.2.

Definition 2.10: Unit Binormal Vector

The **unit binormal vector** for $\mathcal{C}$ is

$$\overrightarrow{B} = \overrightarrow{T} \times \overrightarrow{N}.$$

Then $\overrightarrow{T}$, $\overrightarrow{N}$, and $\overrightarrow{B}$ are three unit vector mutually perpendicular forming the *Frenet frame* (moving coordinate system) along $\mathcal{C}$.

3 Multivariable Functions

3.1 Basics

A multivariable function has domain in $\mathbb{R}^n$ and range in $\mathbb{R}$. That is, $f : \mathbb{R}^n \to \mathbb{R}$.

Example 1

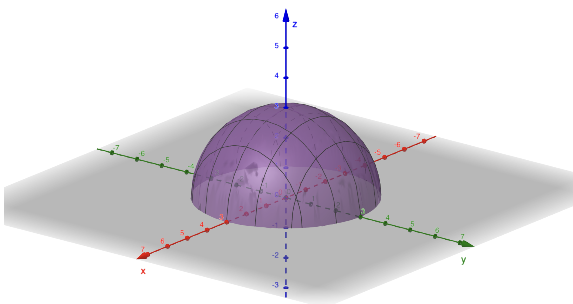
Let $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$. Then $f(3, 2) = \dfrac{\sqrt{6}}{2}$. The domain is (x, y) such that $x = 1$ and $x + y + 1 \geq 0$.

We can graph a function $f(x, y)$ by drawing the domain as xy-plane, and range the z-plane. So the $z = f(x, y)$ is the height above the xy-plane.

Example 2

Graph $g(x, y) = \sqrt{9 - x^2 - y^2}$.

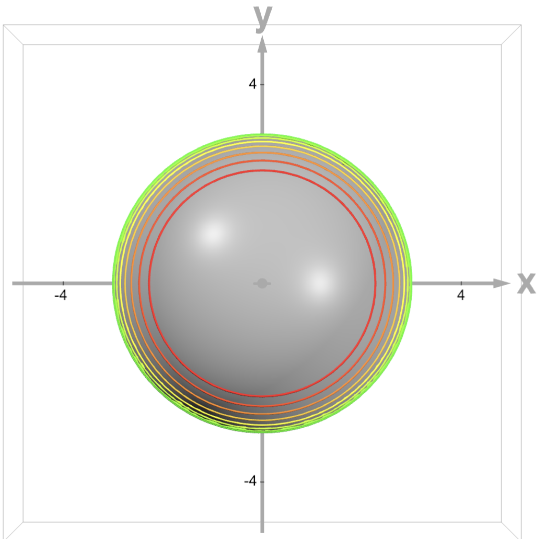
Solution We have $z = \sqrt{9 - x^2 - y^2}$, so $x^2 + y^2 + z^2 = 9$. Since $z \geq 0$, this is the upper half of the sphere with radius 3 and the origin $(0, 0, 0)$.



So the graphing $f(x, y)$ gives the surface of $\mathbb{R}^3$. Then, which surfaces can be graphs of some $f(x, y)$? We use the vertical line test again. If a vertical line hits the surface twice or more, then there are two or more outputs in one input, so it cannot be a graph for some $f(x, y)$.

The contour map of the figure above looks like this:

If labels are evenly spaced, then lines close together indicate steepness.



Functions of Three Variables

The definition is similar to the definition of function of two variables. This function cannot be graphed, since you need 4 dimension to draw. But one still can understand the graph via contour by level surfaces (not level curves).

3.2 Limits and Continuity

Let $f : D \to \mathbb{R}$.

Definition 3.1: Limit

A function $f : \mathbb{R}^n \to \mathbb{R}$ has a **limit** at a point $P \in \mathbb{R}^n$ if $f(x_1, \ldots, x_n)$ becomes arbitrarily close to a single real number, regardless of the path.

Definition 3.2: Continuity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **continuous** at $P \in \mathbb{R}^n$ if

$$\lim_{(x_1,\ldots,x_n)\to P} f(x_1, \ldots, x_n) = f(P).$$

Theorem 3.1

All *standard* functions (polynomials, exponentials, trig, log, etc) are continuous wherever defined.

To compute limits, try plugging in first, and if it doesn't works, try to notice some algebra going on. If none of these work, there is a chance of the limit not existing. To prove that a limit does not exist for some function, you can try plugging in various curves along P. Such curves can be,

- Axes $x = 0$ and $y = 0$
- Lines $y = mx$
- Parabola $y = ax^2$ or $x = ay^2$
- a lot more..

If you ever get two different answers, then the limit does not exist.

Example 3

What is $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$?

Solution Along $y = 0$ we see

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x\to 0} \frac{x(0)^2}{x^2 + 0^4} = 0.$$

Also, along the curve $x = y^2$,

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y\to 0} \frac{(y^2)y^2}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Thus the limit does not exist.

3.3 Partial Derivatives

Recall, in 1 variable functions, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. For 2 variable functions, there are 2 possible inputs to change, so there are 2 different derivatives.

Definition 3.3: Partial Derivatives

The partial derivatives of $f(x, y)$ are

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{(x+h) - x}$$

and

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{(y+h) - y}.$$

Remark.
There is no such thing as $f'(x,y)$.

To compute $\dfrac{\partial f}{\partial x}$, treat other variables as constants, and also for other partial derivatives.

Example 4
Let $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$. Then

$$f_x = \left(\frac{1}{1+y}\right)\cos\left(\frac{1}{1+y}\right) \text{ and } f_y = x(-1)(1+y)^{-2}\cos\left(\frac{x}{1+y}\right).$$

Consider the surface $z = f(x,y)$. Intersecting with the plane $y = b$ will give curve in plane. Then $\left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$ is the slope of this curve at point $\big(a, b, f(a,b)\big)$. In applications, $\dfrac{\partial f}{\partial x}$ measures change in f if we change only x.

Higher-Order Derivatives

We can also compute higher-order derivatives. Notationally, f_{xy} means $\dfrac{\partial}{\partial x}$ then $\dfrac{\partial}{\partial y}$.

Remark.
However, $\dfrac{\partial^2 f}{\partial x \partial y}$ means $\dfrac{\partial}{\partial y}$ then $\dfrac{\partial}{\partial x}$.

Theorem 3.2: Clairaut

If f_{xy} and f_{yx} both exist and are continuous at P, then $f_{xy} = f_{yx}$ at P.

Definition 3.4: Partial Differential Equation

A **partial differential equation** is an equation involving partial derivatives.

3.4 Tangent Plane

For a 2 variable $f(x,y)$, the tangent plane at $(a, b, f(a,b))$ is going to be the plane closest to f at (a,b). We need to find a point and a normal vector. The point is obviously $\overrightarrow{p_0} = \langle a, b, f(a,b)\rangle$, but how do we find the normal vector?

Label the tangent plane $T_{(a,b)}f$. Fix $y = b$. Then the direction vector of the intersection line of $y = b$ and $T_{(a,b)}f$ is $\langle \Delta x, 0, \Delta z\rangle$. Scaling this gives

$$\frac{1}{\Delta x}\langle \Delta x, 0, \Delta z\rangle = \left\langle 1, 0, \frac{\Delta z}{\Delta x}\right\rangle = \langle 1, 0, f_x(a,b)\rangle.$$

Repeating this for $x = a$ gives $\langle 0, 1, f_y(a,b)\rangle$. Since these two vectors are in $T_{(a,b)}f$, the normal vector is

$$\overrightarrow{n} = \langle 0, 1, f_y(a,b)\rangle \times \langle 1, 0, f_x(a,b)\rangle = \langle f_x(a,b), f_y(a,b), -1\rangle.$$

The tangent plane equation is $(\overrightarrow{x} - \overrightarrow{p_0}) \cdot \overrightarrow{n} = 0$, which also is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Remark.

The normal vector $\overrightarrow{n} = \langle f_x, f_y, -1\rangle$ is also orthogonal to the surface $z = f(x,y)$.

Then, for values (x,y) near (a,b), the value z of $T_{(a,b)}f$ will be approximately equal to $f(x,y)$.

Definition 3.5: Linearization

The **linearization** of f at (a,b) is

$$f(x,y) \approx L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Similarly, $f_x dx + f_y dy \approx df$.

Example 5

Suppose there is a cicrular cone with radius 10 and height 25, with possible measurement error by ± 0.1. What is the max error in volume?

Solution The volume is $\frac{1}{3}\pi r^2 h = \frac{2500}{3}\pi$. The error in volume is going to be ΔV, which is

$$\begin{aligned}\Delta V &\approx V_r(10,25)\Delta r + V_h(10,25)\Delta h\\ &= 20\pi.\end{aligned}$$

We have similar formulas for higher dimensions. That is,

$$\begin{aligned}f(x,y,z) &\approx L_{(a,b,c)}(x,y,z)\\ &= f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c),\end{aligned}$$

and so on.

Remark.
The tangent plane approximation works as long as f is differentiable.

Theorem 3.3

f is differentiable at p_0 if f_x and f_y exist and continuous at p_0.

3.5 Chain Rule

In one dimension, we had $\frac{d}{dt}f\big(x(t)\big) = x'(t)f'\big(x(t)\big)$. Now, in $f\big(x(s,t), y(s,t)\big)$, what is f_s?

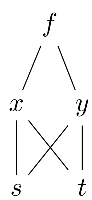
Theorem 3.4: Chain Rule

In $f\big(x(s,t), y(s,t)\big)$,

$$f_s = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

f_t is defined similarly.

We draw the tree diagram to visualize this:



To find f_s, you find all possible routes from f to s, which passes x and y. Then you add up all those product of partial derivatives.

Example 6

Let $x = e^x \sin y$ with $x = st^2$ and $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution We have

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= e^{st^2}\sin(s^2t)\cdot t^2 + e^{st^2}\cos(s^2t)\cdot 2st.\end{aligned}$$

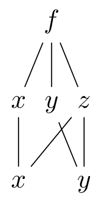
One can also calculate $\frac{\partial z}{\partial t}$ by

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

> **Remark.**
> You should compute answers in terms of the bottom variables (the variables written in the bottom line when drawn the tree diagram) only.

Implicit Differentiation

Suppose $F(x, y, z) = 0$ is an equation that could be solved for $z = f(x, y)$. How do we find $\dfrac{\partial z}{\partial x}$ without solving for z first? The idea is to take $\dfrac{\partial}{\partial x}$ to both sides. The tree diagram gives



Note that x does not depend on y. Only z depends on both x and y. This gives

$$\begin{aligned} \frac{\partial}{\partial x} F(x, y, z) &= \frac{\partial}{\partial x} 0 \\ \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \end{aligned}$$

So

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}.$$

3.6 Directional Derivative and Gradient

In single variable, $f'(x)$ was defined as $\dfrac{\text{change in range}}{\text{unit change in domain}}$. The unit change was in the positive x-direction. However, in multivariable functions, there are many possible direction to move in domain. (Note that f_x is for the positive x-direction, and f_y is for the posiitve y-direction. We need two things:

- Starting point in domain
- Direction of motion in domain.

Definition 3.6: Directional Derivative

Given point $P \in \mathbb{R}^n$ and unit vector $\overrightarrow{u} \in \mathbb{R}^n$. the **directional derivative** of f at point P in direction $\overrightarrow{u}$ is

$$\lim_{h \to 0} \frac{f\left(\overrightarrow{P} + h\overrightarrow{u}\right) - f\left(\overrightarrow{P}\right)}{h}$$

We write as $D_{\overrightarrow{u}} f|_p$.

Of course, we don't want to compute this manually. How do we compute in an easier way?

Definition 3.7: Gradient Vector

Given $f : \mathbb{R}^n \to \mathbb{R}$, the **gradient** of f is the vector

$$\overrightarrow{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots, \right\rangle.$$

Theorem 3.5

$D_{\overrightarrow{u}} f|_p = \overrightarrow{\nabla} f|_p \cdot \overrightarrow{u}$.

Example 7

Let $f(x, y, z) = x \sin(yz)$. Find $D_{\overrightarrow{u}} f|_{(1,3,0)}$ where $\overrightarrow{u}$ is the direction of $\langle 1, 2, -1 \rangle$.

Solution Since $|\langle 1, 2, -1 \rangle| = \sqrt{6}$, $\overrightarrow{u} = \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$. We have

$$\overrightarrow{\nabla} f = \langle \sin yz, xz \cos yz, xy \cos yz \rangle$$
$$\overrightarrow{\nabla} f|_{(1,3,0)} = \langle 0, 0, 3 \rangle,$$

So

$$D_{\overrightarrow{u}} f|_{(1,3,0)} = \langle 0, 0, 3 \rangle \cdot \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle = -\frac{3}{\sqrt{6}}.$$

We now look at the geometry of $\overrightarrow{\nabla} f$.

Theorem 3.6

At point p in domain of f, $\overrightarrow{\nabla} f|_p$ points in direction of greatest increase of f, with magnitude $\left|\overrightarrow{\nabla} f|_p\right|$ the rate of increase.

Example 8

Let the temperature function $T(x, y, z) = \dfrac{80}{1 + x^2 + 2y^2 + 3z^2}$ currently at $(1, 1, -2)$. What is the maximum rate of getting warm?

Solution Calculating gives

$$\overrightarrow{\nabla} T|_p = \left\langle -\frac{5}{8}, -\frac{5}{4}, \frac{15}{4} \right\rangle.$$

Thus the maximum rate of increase is

$$\left|\overrightarrow{\nabla} T|_p\right| = \frac{5}{8}\langle -1, -2, 6\rangle = \frac{5\sqrt{41}}{8}.$$

Theorem 3.7

If we draw a contour map for f, then $\overrightarrow{\nabla} f$ vectors are going to be orthogonal to the level curves (or surfaces).

Theorem 3.8

If surface S in $\mathbb{R}^3$ has equation $F(x, y, z) = k$ constant, then at any $p \in S$, $\overrightarrow{\nabla} F|_p$ is orthogonal to S and the tangent plane of S at p. Thus the equation of the tangent plane of S at p is

$$\overrightarrow{\nabla} F|_p \cdot (\overrightarrow{x} - \overrightarrow{p}) = 0.$$

3.7 Optimization

We now want to find the maximum and minimum points for $f(x, y)$. For single variable functions, in local extremas, f' was 0 or didn't exist.

Definition 3.8: Local Extremum

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has a **local maximum** at $P(a, b)$ in domain $\mathbb{F}$, for all (x, y) near P, $f(P) \geq f(x, y)$. **Local minimum** is defined similarly.

Definition 3.9: Critical Point

A **critical point** $P(a, b) \in \mathrm{Dom}(f)$ is a point where $\overrightarrow{\nabla} f|_p = \overrightarrow{0}$ or does not exist.

Theorem 3.9

If P is a local extrema for f, then P is a critical point.

Remark.

The converse is generally not true. That is, not all critical points are extremas.

Example 9

Find the critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Solution We have $\vec{\nabla} f = \langle 2x - 2, 2y - 6 \rangle$. So $P(1, 3)$.
Then how do we tell which critical points are which?

Definition 3.10: Discriminant

The **discriminant** of $f(x, y)$ is

$$D = f_{xx} f_{yy} - f_{xy}^2 = \det \begin{vmatrix} f_{xx} & f_{yy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Theorem 3.10: Second Derivative Test

Let $P = (a, b)$ be a critical point for f.

- If $D > 0$, then f_{xx} or f_{yy} determines the extrema: If f_{xx} or $f_{yy} > 0$, then f has a local minimum, and if f_{xx} or $f_{yy} < 0$, then f has a local maximum.
- If $D < 0$, then P is a saddle point (something like pringles)
- If $D = 0$, then it is inconclusive.

Example 10

Find and classify the critical points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution The gradient is

$$\vec{\nabla} f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle,$$

So

$$4x^3 - 4y = 0$$

$$4y^3 - 4x = 0.$$

Since $y = x^3$, $x^0 - x = 0$, so this gives $x = 0$ and ± 1. This also gives $y = 0$ and ± 1. The critical points are $(1, 1)$, $(-1, -1)$, and $(0, 0)$. Now, calculating the second partials give

$$f_{xx} = 12x^2,\ f_{yy} = 12y^2,\ \text{and} f_{xy} = f_{yx} = -4,$$

so $D = 144x^2y^2 - 16$. At $(1, 1)$, $D > 0$ and $f_{xx} > 0$, so f has a local minimum at $(-1, -1)$ and its value is -1. At $(0, 0)$, $D < 0$, so it is a saddle point. At $(1, 1)$, $D > 0$ and $f_{xx} > 0$, so f has a local minimum at $(1, 1)$ and its value is -1.

Our goal is to find the global minimum or maximum of f on some domain $\mathcal{D}$.

Definition 3.11: Boundary

In a domain $\mathcal{D}$, $\partial\mathcal{D}$ is used for the **boundary** of $\mathcal{D}$.

The closed interval method and be generalized to 2-variable functions. You will need to find local maximum and minimums within $\mathcal{D}$, then check $\partial\mathcal{D}$.

Example 11

Find the absolute maximum and minimum for $f(x, y) = x^2 - 2xy + 2y$ on a rectangle $\mathcal{D} = \{0 \leq x \leq 3, \leq y \leq 2\}$.

Solution For the interior, $\vec{\nabla} f = \langle 2x - 2y, -2x + 2\rangle$. This gives $(1, 1)$ the critical point with $f(1, 1) = 1$. Divide $\partial\mathcal{D}$ into ∂_1, ∂_2, ∂_3, and ∂_4 where

$$\partial_1 = \{0 \leq x \leq 3, y = 0\}$$

$$\partial_2 = \{0 \leq y \leq 2, x = 3\}$$

$$\partial_3 = \{0 \leq x \leq 3, y = 2\}$$

$$\partial_4 = \{0 \leq y \leq 2, x = 0\}$$

For ∂_1, $f(x, y) = x^2$ where $0 \leq x \leq 3$. So there are two more possible extremum: $(0, 0)$ with $f(0, 0) = 0$ and $(3, 0)$ with $f(3, 0) = 9$. Repeating this, we get several possible more extremum: $f(3, 2) = 1$, $f(2, 2) = 0$, $f(0, 2) = 4$. Therefore, the global minimum is 0 at $(2, 2)$ and $(0, 0)$, and the global maximum is 9 at $(3, 0)$.

Remark.

Note that in the example above, we didn't check for the discriminant at $(1, 1)$. $f(1, 1)$ is just one potential extrema, so we don't need to classify the critical point since we are going to compare them.

Then what if the critical point is the smallest (or largest) of the values compared, but it is a saddle? Fortunately, this won't happen:

Theorem 3.11

For f continuous on $\mathcal{D}$ bounded and closed (containing $\partial\mathcal{D}$), then the absolute extrema exist.

With this, we safely can say that the smallest (or largest) value is not saddle.

What is the boundary is not straight lines? We still want to make the problem 1-dimension-wise. In this case, find a parametrization for $\partial\mathcal{D}$ and then substitute into f. This will reduce the problem into 1-dimension.

Example 12

Let $f(x, y) = x^2 - 2xy + 2y$ with $\mathcal{D}$ the unit disc $\{|(x, y)| \leq 1\}$. In this case, we can use the unit circle parametrization $\overrightarrow{r}(t) = \langle \cos t, \sin t \rangle$. This will reduce f to a single variable function.

3.8 Lagrange Multipliers

Similar with the previous section, our goal is to optimize $f(x, y)$ subject to some constraint $g(x, y) = 0$. The constraint *may not* reduce $f(x, y)$ into 1-variable function. In these case, we use the mehtod of *Lagrange multipliers*.

Theorem 3.12: Lagrange

To find critical points of f subject to $g = 0$, define the auxiliary variable λ and solve the system

$$\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$$

$$g = 0.$$

What is happening geometrically here? We draw $g = 0$ and the level curves of f. Then, $\{g = 0\}$ should be tangential to the level curve at critical points. This gives that the normal vectors for f and g should be parallel, which simplifies to $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$.

Example 13

Optimize $f(x, y) = x^2 + 2y^2$ on $x^2 + y^2 = 1$.

Solution Let $f = x^2 + 2y^2$ and $g = x^2 + y^2 - 1 = 0$. We get

$$\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$$

$$\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle,$$

so a system of equations

$$\begin{aligned} 2x &= 2\lambda x \\ 4y &= 2\lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

First equation gives $x = 0$ or $\lambda = 1$. If $x = 0$, we get $y = \pm 1$. If $\lambda = 1$, we get $y = 0$, and $x = \pm 1$. Therefore, $f(0,1) = f(0,-1) = 2$ is a global maximum, and $f(1,0) = f(-1,0) = 1$ is a global minimum.

Note that the method of Lagrange multipliers is also valid for multiple variables (more that two), and for multiple constraints. In this case, you solve the equation

$$\begin{aligned} \vec{\nabla} f &= \lambda_1 \vec{\nabla} g + \lambda_2 \vec{\nabla} h + \cdots \\ g &= 0 \\ h &= 0 \\ \vdots &= \vdots \end{aligned}$$

With linear algebra knowledge, we say that $\vec{\nabla} f$ is in the span of $\vec{\nabla} g$, $\vec{\nabla} h$, $\ldots$.

4 Multivariable Integration

4.1 Multiple Integrals over Rectangles

In single-variable integration,

$$\int_a^b f\ dx$$

is the area under $y = f(x)$ and above interval $[a, b]$ on the x-axis. The interpretation was to divide the area into an infinitely many 'rectangles'. The base will be dx, and the height will be the f value there. Adding up the areas will give the total area, which is the integral value.

Now, for multiple integration, the notation

$$\iint_{\mathcal{R}} f(x, y)\ dA$$

represents the volume under $z = f(x, y)$ and above region $\mathcal{R}$ on the xy-plane. We now divide the voolume into an infinitely many 'cuboids'. The base is a small xy-square, so $dA = dxdy$. Note that other small shapes are also possible. The height is the f value at that point.

To compute this, we integrate with one variable at a time. That is,

$$\iint_{\mathcal{R}} f\ dA = \int \left(\int f\ dx \right) dy = \int \left(\int f\ dy \right) dx$$

since here, $\displaystyle\int f\ dx$ represents the volume of $x - dy$ slab. You add them up, and vise versa.

If the region is a rectangle $\mathcal{R} = [a, b] \times [c, d]$, then

$$\iint_{\mathcal{R}} f(x, y)\ dA = \int_{y=c}^{d} \left(\int_{x=a}^{b} f\ dx \right) dy = \int_{x=a}^{b} \left(\int_{y=c}^{d} f\ dy \right) dx.$$

Example 1

Evaluate $\displaystyle\iint_{\mathcal{R}} (x - 3y^2)\ dA$ for $\mathcal{R} = [0, 2] \times [1, 2]$.

Solution We have

$$\begin{aligned}\iint_{\mathcal{R}} (x-3y^2)\ dA &= \int_{y=0}^{1} \left(\int_{x=0}^{2} (x-3y^2)dx \right) dy \\ &= \int_{y=1}^{2} \left(\frac{1}{2}x^2 - 3y^2x \right) \Big|_{x=0}^{2} dy \\ &= \int_{y=1}^{2} (2-6y^2)dy \\ &= (2y-2y^3)\Big|_{y=1}^{2} \\ &= -12.\end{aligned}$$

Theorem 4.1: Fubini

For a resonable $f(x,y)$ over any reasonable domain D, $\iint_D f(x,y)\ dA$ can be done x first or y first. The order does not matter.

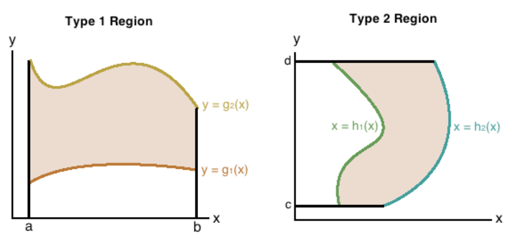
Definition 4.1: Average Value

The **average value** of f on D is

$$f_{\text{avg}} = \frac{1}{\text{Area}(D)} \cdot \iint_D f(x,y)\ dA.$$

4.2 Multiple Integrals over General Regions

We will go over multiple integrals over a general region D, not just rectangles. To do this, we need to understand (or draw) the domain D in xy-plane. (Note that we do not need to draw nor understand the function $z = f(x,y)$. Note that the order now leads to different setups.



Suppose a domain D is bounded by two curves $x = g_1(y)$ and $x = g_2(y)$ meeting at two points (a, b) and (c, d). Then we compute the integral as

$$\iint_D F\ dA = \int_{y=b}^{d} \left[\int_{x=g_1(y)}^{g_2(y)} F(x,y)\ dx \right] dy.$$

If we start with the x part. To start with the y part, we change $x = g_1(y)$ and $x = g_2(y)$ into $y = f_1(x)$ and $y = f_2(x)$ so

$$\iint_D F\ dA = \int_{x=a}^{c} \left[\int_{y=f_2(x)}^{f_1(x)} F(x,y)\ dy \right] dx$$

Example 2

Find the volume under $z = x^2 + y^2$ and above region D bounded by $y = 2x$ and $y = x^2$.

Solution We draw the domain on xy-plane.

so

$$\begin{aligned} \iint_D f\ dA &= \int_{x=0}^{2} \left[\int_{y=x^2}^{2x} (x^2 + y^2)\ dy \right] dx \\ &= \int_{x=0}^{2} \left[x^2 y + \frac{1}{3} y^3 \right]_{y=x^2}^{2x} dx \\ &= \int_{x=0}^{2} \left(2x^3 + \frac{8}{3}x^3 - x^4 - \frac{1}{3}x^6 \right) dx \\ &= \frac{216}{35}. \end{aligned}$$

Volume questions can be tricky. Some may require 3d picture or understanding to see the floor.

> **Example 3**
> Find the volume bounded by the following planes:
>
> $$x + 2y + z = 2$$
>
> $$x = 2y$$
>
> $$x = 0$$
>
> $$z = 0$$

Solution To see the floor, we need to find the line $x + 2y + z = 2$ intersects with the xy-plane. We set $z = 0$, so $x + 2y = 2$. Thus the domain is the region enclosed by $x = 2y$, $x + 2y = 2$, and $x = 0$. Thus the volume is

$$\iint_D (2 - x - 2y)\ dA = \int_{x=0}^{1} \int_{y=x/2}^{y=(2-x)/2} (2 - x - 2y)\ dydx,$$

which can be calculated.

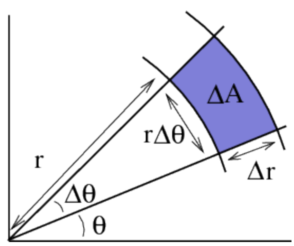
Sometimes a double integral is impossible one way, so you will need to switch orders. Draw the domain first!

4.3 Double Integrals in Polar Coordinates

Recall that the polar coordinates is written (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$. This will give $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.

There are two ways to compute the integral: either r first or θ first. If we integrate with r first, we get radial slabs, and if we integrate with θ first, we get circular arc slabs.

No matter of the order, we need to have an understanding of the small bit of area dA. To compute the area of this rectangle-ish (since it is tiny) figure, the change in radius is dr, and the change in arc length is $rd\theta$.



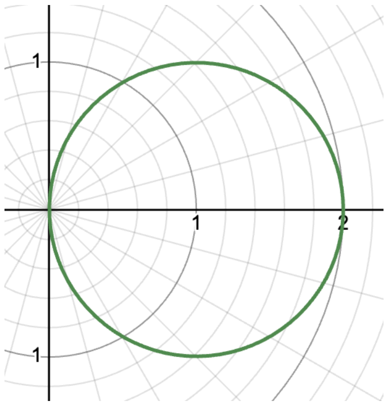
So

$$dA = rdrd\theta.$$

Example 4

Find the volume under $z = x^2 + y^2$, above xy-plane, within the cylinder $x^2 + y^2 = 2x$.

Solution Note that $x^2 + y^2 = 2x$ is equivalent to $r^2 = 2r\cos\theta$, so $r = 2\cos\theta$. We integrate with r first. The r should start from 0 and end at $2\cos\theta$.



Since θ varies from $-\pi/2$ to $\pi/2$, we get

$$\begin{aligned}
\text{Volume} &= \iint_D (x^2 + y^2)\ dA \\
&= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot rdrd\theta \\
&= \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{4} \cdot (2\cos\theta)^4\right] d\theta \\
&= \int_{\theta=-\pi/2}^{\pi/2} 4\cos^4\theta\ d\theta \\
&= \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin^4\theta\right]_{\theta=-\pi/2}^{\pi/2} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

4.4 Basic Applications

Recall that $\iint_D f\ dA$ is the (signed) volume under $z = f(x, y)$, above D. We have some basic applications:

- $\iint_D (f - g)dA$ is the volume between f and g on D.
- $\iint_D 1\ dA$is the area of D.

Definition 4.2: Density Function

A **density function** is a function designed so that integrating the function will give the total amount.

We have the mass density $\rho(x, y)$ and charge density $\sigma(x, y)$ as two examples. One important example is a probability density. This $f(x, y)$ means that you are taking measurements of two things and $\iint_D f\ dA$ is the probability that your measurements lie within the shape D on the xy-plane.

4.5 Triple Integration

The triple integral

$$\iiint_E f\ dV$$

represents the 4-dimension volume under $w = f(x, y, z)$, above shape E on the xyz-space. Note that it is impossible to visualize the function. Thus we ignore the 4-dimensional shape and just understand the shape of E on the 3-dimensional floor.

The easiest case is the cuboid $E = [a, b] \times [c, d] \times [e, f]$. Then the integral is simply

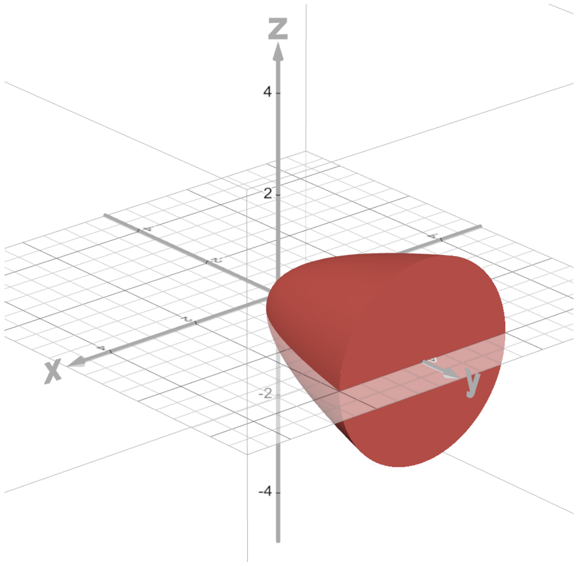
$$\iiint_E f\ dV = \int_{z=e}^{f} \int_{y=c}^{d} \int_{x=a}^{b} f(x, y, z)\ dxdydz$$

where the orders can be swapped. For other shapes, you do the same thing: compute one integral first, and this will collapse the problem into a double integral, which we are already familiar of. While computing the remaining double integral, draw the domain again to clarify where you're integrating over.

Example 5

Compute $\iiint_E \sqrt{x^2 + z^2} dV$ where E is the region bounded by paraboloing $y = x^2 + z^2$ and plane $y = 4$.

Solution We first understand how E looks like.



If doing z first, then this will give

$$\iint_D \int_{z==\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdA$$

which is difficult, so we will try other variable first. Trying y first gives

$$\iiint_E \sqrt{x^2+y^2}\ dV = \iint_D \int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\ dydA$$

For the remaining double integral, the domain is $x^2+z^2=4$. We now switch to polar coordinates. This gives

$$\begin{aligned} iint_D \int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\ dydA &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{y=r^2}^{4} r\ dy\ rdrd\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4r^2 - r^4)\ drd\theta \\ &= \int_{\theta=0}^{2\pi} \frac{64}{15}\ d\theta \\ &= \frac{128}{15}\pi. \end{aligned}$$

Note that all 2-dimensional applications still apply here:

- The average value of f is $\dfrac{1}{\text{Vol}(E)} \iiint_E f\ dV$

- $\iiint_E 1\ dV = \text{Vol}(E)$
- $\iiint_E \text{Density}(E) = \text{Total within } E.$

4.6 Cylindrical Coordinates

The above example can be interpreted as cylindrical coordinates, (r, θ, z), where the xy-plane is treted with polar coordinates. We use

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z.$

Example 6

Draw $z = r$.

Solution The first solution is to convert back into $z = \pm\sqrt{x^2+y^2}$, so $z^2 = x^2+y^2$. This is a cone.

If you plug in several values and think about it, it will give you the cone.

In cylindrical coordinates, you use $r\ dzdrd\theta$, up to any order.

Example 7

Solid E within cylinder $x^2 + y^2 = 1$, below plane $z = 4$, above paraboloid $z = 1 - x^2 - y^2$. Density at any point is proportional to distance from the z-axis. Find the total mass.

Solution The density is $\rho = k \cdot r$. So the mass is

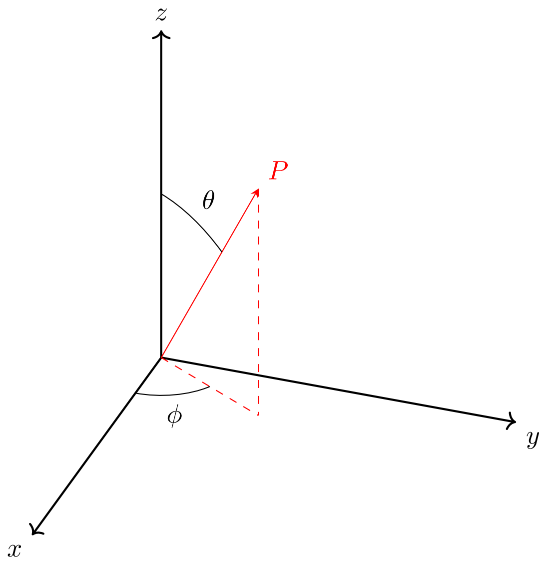
$$\iiint_E kr\ dV = k\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r^2}^{4} r^2\ dzdrd\theta$$

which can be calculated, and this is $12k\pi/5$.

4.7 Spherical Coordinates

This is the *real* or *new* 3-dimension polar coordinates. Given a point $P(\rho, \theta, \phi)$,

1. Stand at origin facing x-axis direction with arm point straight up (z-axis).
2. Turn your body to face P. This turning angle is θ.
3. Decline your arm to point at P. This declination angle is ϕ.
4. Stretch your arm to reach P. This distance from the origin is ρ.



Then we have the following conversions:

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

and $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = y/x$, $\cos\phi = z/\rho = z/\sqrt{x^2 + y^2 + z^2}$.

This gives the following key equations:

- If ρ is a constant, then P is on the sphere of radius ρ with center the origin.
- If θ is a constant, then P is the plane on the constant θ-direction.
- If ϕ is a constant, then P is on the cone

While integrating, we use $dV = \rho^2 \sin\phi \; d\rho d\phi d\theta$, where the integration can be done in any order even though you typically do ρ first.

Remark.

θ will at most vary from 0 to 2π, but ϕ will usually vary from 0 to ϕ.

Example 8

Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \; dV$ for B the unit ball $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$.

Solution The region is $\rho \leq 1$. Thus

$$\begin{aligned}\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\ dV &= \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} e^{\rho^3}\rho^2\sin\phi\ d\rho d\theta d\phi \\ &= \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \frac{1}{3}e^{\rho^3}\Big|_0^1 \sin\phi\ d\theta d\phi \\ &= \frac{1}{3}(e-1)\cdot 2\pi\cdot 2 \\ &= \frac{4}{3}(e-1)\pi.\end{aligned}$$

> **Example 9**
> Find the volume above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

Solution Note that $z = \sqrt{x^2+y^2}$ is a cone and the sphere $x^2+y^2+z^2 = z$ is a sphere centered at $(0, 0, 1/2)$ with radius $1/2$.

Converting gives $z^2 = x^2+y^2$ to $\phi = \pi/4$ and $x^2+y^2+z^2 = z$ to $\rho = \cos\phi$. This gives the converted domain

$$D = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq \cos\phi, 0 \leq \phi \leq \pi/4, 0 \leq \theta \leq 2\pi\}.$$

Thus

$$\begin{aligned}\text{Volume} &= \iiint_D 1\ dV \\ &= \int_{\phi=0}^{\pi/4}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{\cos\phi} 1\cdot\rho^2\sin\phi\ d\rho d\theta d\phi \\ &= \frac{\pi}{8}.\end{aligned}$$

4.8 Change of Variables

In single variable integration, we take u-substitutions so that

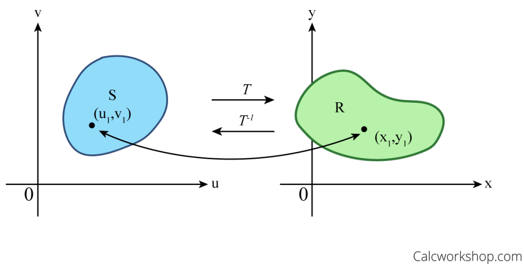
$$\int_a^b f(x)\ dx = \int_{F(a)}^{F(b)} f\frac{1}{F(x)}\ du$$

with $u = F(x)$. Geometrically, we are finding the area of $f(x)$ under the interval $I_x = [a, b]$, and this is changed to finding the area f/F under the interval $I_u = F(I_x) = [F(a), F(b)]$. Let G be the inverse function of F. Then really

$$\int_{I_x} f(x)\ dx = \int_{F(I_x)} f\cdot G'\ du.$$

When integrating, you have an option to change the domain. This is also valid for multiple integrals.

In a two-variable function, if we have some $\iint_R f(x,y)\,dA$ where R is some weird region, we want to change R to some "easy" region to handle. Let this transformed region be S, and $R = T(S)$ so that $S = T^{-1}(R)$. Note that the boundary of R is mapped to the boundary of S (this is because a continuous image of a compact set is also compact, but not to worry here). Then



$$\iint_R f\,dA = \iint_S f\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA$$

with the conversion factor

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|.$$

This value is called the *Jacobian*.

Example 10

Let $T(u,x) = \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$. If S is a unit square, find $T(S)$.

Solution We divide S into four parts, S_1, S_2, S_3, and S_4, where

$$S_1 = \{(u,v) \mid 0 \le u \le 1, v = 0\}$$

$$S_2 = \{(u,v) \mid u = 1, 0 \le v \le 1\}$$

Then $T(S_1) = \{(x,y) \mid 0 \le x \le 1, y = 0\}$.

For S_2, we have $x = 1 - v^2$ and $y = 2v$ for $0 \le v \le 1$. So $x = 1 - y^2/4$ with $0 \le y \le 2$. So $T(S_2) = \{(x,y) \mid x = 1 - y^2/4, 0 \le y \le 2\}$. Doing the same process for S_3 and S_4, we have

$$T(S_3) = \{(x, y) \mid x = -1 + y^2/4, 0 \le y \le 2\}$$
$$T(S_4) = \{(x, y) \mid -1 \le x \le 0, y = 0\}.$$

So we have $R = T(S) = T(S_1) \cup T(S_2) \cup T(S_3) \cup T(S_4)$.

Example 11

Compute $\iint_R y \, dA$ for R as in the previous example.

Solution The Jacobian is

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (2u) \cdot (2u) - (-2v) \cdot (2v) = 4u^2 + 4v^2.$$

So

$$\begin{aligned}\iint_R y \, dA &= \iint_S (2uv) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \, dA \\ &= \int_{v=0}^{1} \int_{u=0}^{1} (2uv)(4u^2 + 4v^2) \\ &= 2\end{aligned}$$

Note that if more natural to define T^{-1}, then one needs to solve for x and y to understand T.

Example 12

Compute $\iint_R e^{(x+y)/(x-y)}$ where R is the trapezoid (diagram shit how do i draw this)

Solution solve this integral shit

Example 13

Let R be the region bounded by $y = 2x - 1$, $y = 2x + 1$, $y = 1 - x$, and $y = 3 - x$. Find a good T^{-1} so that $S = T^{-1}(R)$ is a rectangle.

Solution Changing the bounds' equations give $y = 2x = \pm 1$ and $y + x = 1, 3$. So let $u = y - 2x$ and $v = y + x$. This gives the rectangle $[-1, 1] \times [1, 3]$ on the uv-plane.

5 Vector Calculus

In the previous chapters, we studied vector valued functions $\overrightarrow{r}(t) : \mathbb{R} \to \mathbb{R}^n$, and multivariable functions $f : \mathbb{R}^n \to \mathbb{R}$. Now, we study functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

5.1 Vector Fields

Definition 5.1: Vector Field

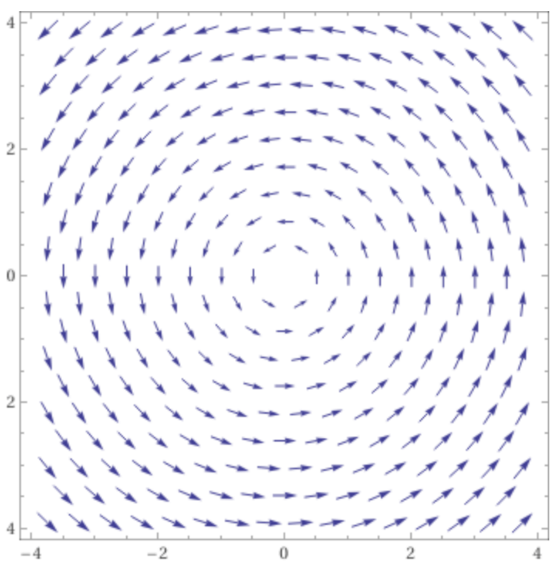
A **vector field** is a function $\overrightarrow{F}(x, y, z) : \mathbb{R}^n \to \mathbb{R}^n$ which maps (x, y, z) into a vector $\langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

We visualize vector fields by plotting some vectors at points in the domain.

Example 14

Draw $\overrightarrow{F}(x, y) = \langle -y, x\rangle$.

Solution



Remark.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function, then $\overrightarrow{\nabla} f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field.

Example 15

If $f(x,y) = x^2y - y^3$, then $\overrightarrow{\nabla} f = \langle 2xy, x^2 - 3y^2 \rangle$ is a vector field.

Definition 5.2: Conservative Vector Field, Potential Function

A vector field $\overrightarrow{F}$ is called **conservative** if $\overrightarrow{F}$ is the gradienf of some f. If so, f is called a **potential function** for $\overrightarrow{F}$.

Remark.

There may be more than one potential functions (including constant differences).

Force Fields

Vector fields can model *velocity fields* for motion/flows or *force fields* illustrating the force one would feel at each point.

Definition 5.3: Gravitational Force Field

The **gravitational force field** at the point (x,y,z) is defined by

$$\overrightarrow{F}(x,y,z) = -\frac{MmG}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z \rangle$$

and is the force felt by mass M if positioned at the point (x,y,z).

Notice that the gravitational force field is convervative and has potential

$$f(x,y,z) = \frac{MmG}{(x^2+y^2+z^2)^{1/2}}.$$

5.2 Operations on Vector Fields

We use the notation $\overrightarrow{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$, so that $\overrightarrow{\nabla} f = \langle f_x, f_y, f_z \rangle$.

Definition 5.4: Divergence

The **divergence** of vector field $\overrightarrow{F} = \langle P, Q, R \rangle$ is defined as

$$\text{div}\, \overrightarrow{F} = \overrightarrow{\nabla} \cdot \overrightarrow{F} = P_x + Q_y + R_z.$$

Example 16

$\text{div}\langle xz, xyz, -y^2 \rangle = z + xz + 0.$

The divergence at point P measures *outward flow* from P (if it has inward flow, then the divergence is negative). If the divergence is zero (globally), we call $\overrightarrow{F}$ **incompressible**.

Definition 5.5: Curl

The **curl** of $\overrightarrow{F} = \langle P, Q, R \rangle$ is

$$\operatorname{curl} \overrightarrow{F} = \overrightarrow{\nabla} \times \overrightarrow{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

The curl at point P measures the tendency of fluid to *rotate* about P, and the direction of the curl is perpendicular to the plane of rotation.

So overall,

$$\text{Scalar functions} \underset{\overrightarrow{\nabla}}{\longrightarrow} \text{Vector fields} \underset{\text{curl}}{\longrightarrow} \text{Vector fields} \underset{\text{div}}{\longrightarrow} \text{Scalar functions}$$

Theorem 5.1

- $\operatorname{curl}\left(\overrightarrow{\nabla} f\right) = \overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} f\right) = \overrightarrow{0}$
- $\operatorname{div}\left(\operatorname{curl} \overrightarrow{\nabla} f\right) = \overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla} f \times \overrightarrow{F}\right) = 0.$

So, any two in a row (in order) is zero.

Remark.
If $\overrightarrow{F}$ is conservative, then $\operatorname{curl} \overrightarrow{F} = \overrightarrow{0}$, and if $\operatorname{curl} \overrightarrow{F} \neq \overrightarrow{0}$, then $\overrightarrow{F}$ is not conservative.

Theorem 5.2

If $\operatorname{curl} \overrightarrow{F} = \overrightarrow{0}$ and the domain of $\overrightarrow{F}$ is simply connected then $\overrightarrow{F}$ is conservative.

Note that in $\mathbb{R}^2$, $\overrightarrow{F} = \langle P, Q \rangle$, we can interpret $\operatorname{curl} \overrightarrow{F}$ as the three dimensional vector with z component zero. Then

$$\overrightarrow{\nabla} \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle,$$

so it is enough to check $P_y = Q_x$. The word *simply connected* in $\mathbb{R}^2$ means basically no holes.

Example 17
In $\mathbb{R}^2$, the upper-half plane is simply connected, but if D is all $\mathbb{R}^2$ except the origin, then D is not simply connected.

Then, what actually does simply connected mean?

Definition 5.6: Simply Connected Domain

A domain in $\mathbb{R}^n$ is **simply connected** if drawn any loop $\mathcal{L}$ in D, then it is possible to deform $\mathcal{L}$ to a point without leaving D.

Example 18

$\mathbb{R}^3$ except the origin is simply connected, but $\mathbb{R}^3$ except the z-axis is not simply connected. The solid ball with radius 2 with radius 1 ball removed is simply connected, but the solid torus (doughnut) is not simply connected.

Example 19

Let $\overrightarrow{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ Since curl $\overrightarrow{F} = \overrightarrow{0}$ and the domain is $\mathbb{R}^3$ which is simply connected, $\overrightarrow{F}$ is conservative. In face,

$$\overrightarrow{F} = \overrightarrow{\nabla}\langle xy^2z^3\rangle.$$

Definition 5.7: Laplacian

The **laplacian** of a scalar function f is

$$\overrightarrow{\nabla}^2 f = \text{div}\left(\overrightarrow{\nabla} f\right) = f_{xx} + f_{yy} + f_{zz}.$$

5.3 Line Integration

Our goal here is to integrate various types of functions over 1-dimension domain (curve $\mathcal{C}$) in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Definition 5.8: Line Integral

Suppose $f(x, y)$ is a scalar function in $\mathbb{R}^2$ (or $f(x, y, z)$ in $\mathbb{R}^3$). The **scalar line integral** of f along $\mathcal{C}$ is written $\int_{\mathcal{C}} f\ ds$ and computed as

$$\int_{\mathcal{C}} f\ ds = \int_{t=a}^{b} f\left(\overrightarrow{r}(t)\right)|\overrightarrow{r}'(t)|\ dt.$$

Recall that s is the arc length function. So $ds = |\overrightarrow{r}'(t)|dt$ and this seems true since $\frac{ds}{dt} = |\overrightarrow{r}'(t)|$.

Example 20

Compute $\int_{\mathcal{C}} (2 + x^2y)\ ds$ where $\mathcal{C}$ is the upper half of the unit circle.

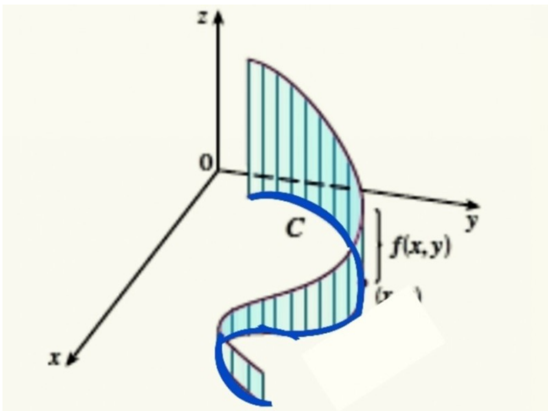
Solution Let $\overrightarrow{r}(t) = \langle \cos t, \sin t \rangle$, where $t : 0 \to \pi$. Then $\overrightarrow{r}'(t) = \langle -\sin t, \cos t \rangle$, so $|\overrightarrow{r}'(t)| = 1$. Thus

$$\begin{aligned}\int_{\mathcal{C}} (2 + x^2 y)\ ds &= \int_{t=0}^{\pi} \left[2 + \cos^2 t \sin t\right] \cdot 1\ dt \\ &= 2\pi + \frac{2}{3}.\end{aligned}$$

Some properties are:

- $\int_{\mathcal{C}} 1\ ds$ is the arc length of $\mathcal{C}$
- If $\mathcal{C}$ is a thin wire, then $\int_{\mathcal{C}} (\text{density})\ ds$ is the total value.

Geometrically, $\int_C f\ ds$ the bent area above $\mathcal{C}$.



Computing $\int_{\mathcal{C}} f\ dx$ and $\int_{\mathcal{C}} f\ dy$ are the same procedure, but instead of $ds = |\overrightarrow{r}'(t)|\ dt$, we use $dx = x'(t)\ dt$ and $dy = y'(t)\ dt$.

Definition 5.9: Vector Line Integral

If $\overrightarrow{F}(x, y, \dots)$ is a vector field in $\mathbb{R}^n$, the **vector line integral** of $\overrightarrow{F}$ along an *oriented curve* $\mathcal{C}$ is

$$\int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{t=a}^{b} \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\ dt.$$

The *oriented curve* $\mathcal{C}$ requires parametrization $\overrightarrow{r}(t)$ for $\mathcal{C}$ that respects the orientation (travels in correct direction).

Example 21

Find $\int_{\mathcal{C}} \langle xy, yz, xz\rangle \cdot d\overrightarrow{r}$ over $\mathcal{C}$ given by $x = t$, $y = t^2$, and $z = t^3$ where $0 \leq t \leq 1$.

Solution Since $\overrightarrow{r}(t) = \langle t, t^2, t^3\rangle$, $\overrightarrow{r}'(t)\langle = 1, 2t, 3t^2\rangle$. So our integral is

$$\int_{t=0}^{1} \langle t^3, t^5, t^4\rangle \cdot \langle 1, 2t, 3t^2\rangle \; dt = \int_{t=0}^{1} (5t^6 + t^3) \; dt = \frac{27}{28}.$$

Remark.

If we get the orientation backwards, we will negate the answer.

If we view $\overrightarrow{F}$ as a force, then $\int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r}$ is the work done by $\overrightarrow{F}$ along $\mathcal{C}$, which can be interpreted as "how much $\overrightarrow{F}$ helped push along $\mathcal{C}$ overall". So, if given a field $\overrightarrow{F}$ and the curve $\mathcal{C}$ with orientation, it is enough to look at the picture to figure out the sign of $\int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r}$.

Remark.

For piecewise smooth curves, integrate one piece at a time.

5.4 Fundamental Theorem of Line Integration

The fundamental theorem of calculus told us that

$$\int_a^b f'(x) \; dx = f(b) - f(a).$$

Theorem 5.3: Fundamental Theorem of Line Integration

If $\mathcal{C}$ is (smooth) curve from A to B (not necessarily distinct), and $f(x, y, \dots)$ is a scalar function, then

$$\int_{\mathcal{C}} \overrightarrow{\nabla} f \cdot d\overrightarrow{r} = f(B) - f(A).$$

Remark.

The actual curve $\mathcal{C}$ is irrelevant, and only endpoints matter.

So if $\overrightarrow{F}$ is conservative, then

$$\int_{\mathcal{C}_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{\mathcal{C}_1} \overrightarrow{F} \cdot d\overrightarrow{r}$$

as long as curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have same endpoints. This is called *path independence.*

Corollary

If $\mathcal{C}$ is a closed path, then

$$\oint_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r} = 0.$$

Note that $\oint$ means an integral along a closed loop.

More generally, if curl $\overrightarrow{F} = \overrightarrow{0}$ on some non-simply connected domain D, then

$$\int_{\mathcal{C}_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{\mathcal{C}_2} \overrightarrow{F} \cdot d\overrightarrow{r}$$

as long as $\mathcal{C}_1$ can be deformed into $\mathcal{C}_2$ without leaving D.

One question is: how do we find f making $\overrightarrow{F} = \overrightarrow{\nabla} f$ where $\overrightarrow{F}$ is a given vector field? We take partial integrations. This is best explained with an example.

Example 22

Find a potential $g(x, y)$ so

$$\overrightarrow{G} = \langle 3 + 2xy, x^2 - 3y^2 \rangle = \overrightarrow{\nabla} g.$$

Solution We want g such that

$$\frac{\partial g}{\partial x} = 3 + 2xy$$
$$\frac{\partial g}{\partial y} = x^2 - 3y^2.$$

We integrate $3 + 2xy$ with respect to x, which gives

$$g(x) = \int 3 + 2xy \; dx = 3x + x^2 y + \phi(y).$$

Note that the integration constant is a function of y instead of just a constant, since we integrated with respect to only x. Now,

$$\frac{\partial g}{\partial y} = x^2 + \phi'(y)$$

So $\phi'(y) = -3y^2$, and $\phi(y) = -y^3 + c$ for some constant c. Now this c is only a constant since ϕ is a function only depending on y. Thus $g(x, y) = 3x + x^2 y - y^3 + c$.

5.5 Green's Theorem

Theorem 5.4: Green

If $\overrightarrow{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field in $\mathbb{R}^2$, then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} \overrightarrow{F} \cdot d\overrightarrow{r}$$

where ∂D is traversed in positive orientation.

Note that $\overrightarrow{F} \cdot d\overrightarrow{r} = P\,dx + Q\,dy$. The meaning of positive orientation of a boundary of a curve is in a way such that D is to the left, which is typically counterclockwise. We use the symbol $\oint$ to note positive orientation. Green's theorem is often used to change line integrals to double integrals and then compute.

Example 23

Compute $\oint_{\mathcal{C}} \langle x^4, xy\rangle \cdot d\overrightarrow{r}$ where $\mathcal{C}$ is the triangle passing $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(0, 0)$ again in order.

Solution Let D be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. We have

$$\oint_{\mathcal{C}} \langle x^4, xy\rangle \cdot d\overrightarrow{r} = \iint_D y\ dA = \int_{x=0}^{1} \int_{y=0}^{1-x} y\ dydx = \frac{1}{6}.$$

In $\mathbb{R}^2$, if $\overrightarrow{F}$ is conservative, then $P_y = Q_x$. Then Green gives

$$\oint_{\partial D} \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_D 0\ dA = 0,$$

which is the statement of fundamental theorem of line integration for closed curves.

Example 24

Evaluate $\oint_{\mathcal{C}} \left\langle 3y - e^{\sin x}, 7x + \sqrt{y^4 + 1}\right\rangle \cdot d\overrightarrow{r}$ where $\mathcal{C}$ is the circle $x^2 + y^2 = 9$.

Solution We have

$$\oint_{\mathcal{C}} \left\langle 3y - e^{\sin x}, 7x + \sqrt{y^4 + 1}\right\rangle \cdot d\overrightarrow{r} = \iint_D (7 - 3)\ dA = 36\pi.$$

Corollary

$$\text{area}(D) = \oint_{\partial D} x\ dy = -\oint_{\partial D} y\ dx = \frac{1}{2}\oint \langle -y, x\rangle \cdot d\vec{r}.$$

Example 25

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution Note that the ellipse can be parametrized by $\langle x, y\rangle = \langle a\cos t, b\sin t\rangle$. So

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\oint \langle -b\sin t, a\cos t\rangle \cdot \vec{r}\,'(t)\ dt \\
&= \frac{1}{2}\int_{t=0}^{2\pi} \langle -b\sin t, a\cos t\rangle \cdot \langle -a\sin t, b\cos t\rangle\ dt \\
&= \frac{1}{2}\int_{t=0}^{2\pi} ab\ dt \\
&= ab\pi.
\end{aligned}$$

If you integrate along a boundary which is the opposite direction of the positively orientated direction, then we introduce a minus sign: $\mathcal{C} = -\partial D$. This will also give a minus sign to the final value of the integral.

Remark.

If D has holes, then ∂D can be multiple curves, each with its own orientation requirements.

5.6 Parametric Surfaces

A parametric surface functon $\vec{r}(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$. Each (u, v) corresponds to a point on a surface $\mathcal{S}$ of a solid.

Example 26

Let $\vec{r}(u, v) = \langle 2\cos u, v, 2\sin u\rangle$ with domain $\mathbb{R}^2$. What is the surface?

Solution Note that $x^2 + z^2 = 4$ and y could be anything, so it is a cylinder.

Note that the boundary of a parametric surface is the image of the boundary of the domain. Thus the surface in the example above, there is no boundary.

Remark.

If given a surface, it's difficult to figure out $\vec{r}(u, v)$.

So we have some tricks.

For planes, choose a point $\overrightarrow{P}$ on the plane and two non-parallel vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ that are both parallel to the plane. Then

$$\overrightarrow{r}(u,v) = \overrightarrow{P} + u\overrightarrow{a} + v\overrightarrow{b}.$$

If a surface has equation of the form

$$\text{one variable} = f(\text{two other variables}),$$

then you let u and v to be the two other variables in the function above.

> **Example 27**
>
> Find a parametrization for a paraboloid $z = x^2 + 2y^2$.

Solution Let $u = x$ and $v = y$ gives $\overrightarrow{r}(u,v) = \langle u, v, u^2 + 2v^2\rangle$.
If a surface is of the form

$$f(\text{two variables}) = \text{constant},$$

this function in $\mathbb{R}^2$ will be a curve $\mathcal{C}$. First find the parametrization $\overrightarrow{r}(u)$ of this curve, and letting v the free variable will give you the parametrization $\overrightarrow{r}(u,v)$.

> **Example 28**
>
> If given $x^2+z^2 = 4$, in the xz-plane, this is a circle, so $\overrightarrow{r}(u) = \langle 2\cos u, 2\sin u\rangle$. Letting $v = y$ gives $\overrightarrow{r}(u) = \langle 2\cos u, v, 2\sin u$.

> **Example 29**
>
> Find a parametrization for a sphere of radius a centered at the origin.

Solution Using spherical coordinates gives $\overrightarrow{r}(u,v) = \langle a\sin v\cos u, a\sin v\sin u, a\cos v\rangle$.

> **Remark.**
>
> One can have multiple parametrizations for a surface.

Note that $\overrightarrow{r}_u = \dfrac{\partial \overrightarrow{r}}{\partial u}$ give a vector tangent to $\mathcal{S}$ in u-direction, and $\overrightarrow{r}_v = \dfrac{\partial \overrightarrow{r}}{\partial v}$ give a vector tangent to $\mathcal{S}$ in v-direction. Thus $(\overrightarrow{r}_u \times \overrightarrow{r}_v \perp \mathcal{S}$, and this can be used to find the tangent plane $T_P\mathcal{S}$.

> **Example 30**
>
> Find the tangent plane to $\mathcal{S}$ parametrized by $\overrightarrow{r}(u,v) = \langle u^2, v^2, u + 2v\rangle$ at $P(1, 1, 3)$.

Solution We have $\vec{r}_u = \langle 2u, 0, 1\rangle$ and $\vec{r}_v = \langle 0, 2v, 2\rangle$. At $P(1,1,3)$ $u = 1$ and $v = 1$, so we have the normal vector $\langle 2,0,1\rangle \times \langle 0,2,2\rangle = \langle -2,-4,r\rangle$. Thus the tangent plane is $-2x - 4z + 4z = 6$.

Definition 5.10: Surface Area

If D is the domain o $\vec{r}(u,v)$ for $\mathcal{S}$, and $\vec{r}$ is one-to-one, then the **surface area** of $\mathcal{S}$ is

$$\text{Area}(\mathcal{S}) = \iint_D |\vec{r}_u \times \vec{r}_v| \; dA.$$

Example 31

Find the area of $z = x^2 + y^2$ under $z = 9$.

Solution We have $\vec{r}(u,v) = \langle u, v, u^2 + v^2\rangle$, so $\vec{r}_u = \langle 1, 0, 2u\rangle$ and $\vec{r}_v = \langle 0, 1, 2v\rangle$. Thus $\vec{r}_u \times \vec{r}_v = \langle -2u, -2v, 1\rangle$, so $|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + 4u^2 + 4v^2}$. Since the domain is $u^2 + v^2 = 9$ (or $r = 3$ in polar coordinates), so

$$\iint_D \sqrt{1 + 4u^2 + 4v^2} \; dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r\sqrt{1 + 4r^2} \; dr d\theta = \frac{\pi}{6}(37\sqrt{37} - 1).$$

5.7 Surface Integration

Definition 5.11: Scalar Surface Integral

The **scalar surface integral** of a function $f(x,y,z)$ over a surface $\mathcal{S}$ with parametrization $\vec{r}(u,v)$ (with domain D)

$$\iint_{\mathcal{S}} f \; d\mathcal{S} = \iint_D f\left(\vec{r}(u,v)\right) |\vec{r}_u \times \vec{r}_v| \; dA$$

Here, $d\mathcal{S} = |\vec{r}_u \times \vec{r}_v| \; dA$.

Remark.

To compute a surface integral, one needs a parametrization for $\mathcal{S}$.

Example 32

Compute $\iint_{\mathcal{S}} x^2 \; d\mathcal{S}$ for $\mathcal{S}$ the unit sphere.

Solution We have $\vec{r}(u,v) = \langle \cos u \sin v, \sin u \sin v, \cos v\rangle$ with $D = \{0 \leq v \leq$

$2\pi, 0 \leq v \leq \pi\}$, and

$$\begin{aligned} \vec{r}_u &= \langle -\sin u \sin v, \cos u \sin v, 0\rangle \\ \vec{r}_v &= \langle \cos u \cos v, \sin u \cos v, -\sin v\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v. \end{aligned}$$

So

$$|\vec{r}_u \times \vec{r}_v| = |\sin v| = \sin v$$

since $0 \leq v \leq \pi$. Thus

$$\iint_{\mathcal{S}} x^2 \, d\mathcal{S} = \int_{v=0}^{\pi} \int_{u=0}^{2\pi} \cos^2 v \sin^2 v \cdot \sin v \, dudv = \frac{4}{3}\pi.$$

If instead we had $\pi/2 \leq v \leq 3\pi/2$ so that the absolute value of $\sin v$ atttain both positive and negative values, you split the absolute value function into a piecewise function

$$|\sin v| = \begin{cases} \sin v & \pi/2 \leq v \leq \pi \\ -\sin v & \pi \leq v \leq 3\pi/2 \end{cases}.$$

What did $\iint_{\mathcal{S}} f \, d\mathcal{S}$ compute? This can be interpreted as the *bent volume* where $\mathcal{S}$ is the bottom plane. If you integrate a density function over a surface $\mathcal{S}$, you get the total amount for $\mathcal{S}$.

Definition 5.12: Oriented Surface

An **oriented surface** $\mathcal{S}$ is one with two possible choice of normal vector $\vec{n} \perp \mathcal{S}$. If $\mathcal{S}$ allows such a choice, $\mathcal{S}$ is called **orientable**.

Example 33

The mobius strip is not orientable!

Definition 5.13: Vector Surface Integral

The **vector surface integral** of $\vec{F}(x, y, z)$ over an oriented $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{\mathcal{S}} = \pm \iint_{D} \vec{F}\left(\vec{r}(u, v)\right) \cdot \left(\vec{r}_u \times \vec{r}_v\right) \, dA$$

where the sign of $\pm$ is determined by the orientation:

- $+$ if $\vec{r}_u \times \vec{r}_v$ is the same direction as $\vec{n}$
- $-$ if $\vec{r}_u \times \vec{r}_v$ is the opposite direction as $\vec{n}$

Remark.
The vector surface integrals are only defined for oriented surfaces.

In physics, $\iint_{\mathcal{S}} \overrightarrow{F} \cdot d\overrightarrow{\mathcal{S}}$ is also called the *flux* of $\overrightarrow{F}$ across $\mathcal{S}$.

- If parametrization $\overrightarrow{r}(u,v)$ is given and $\overrightarrow{n}$ is not specified, then one can assume that $\overrightarrow{n}$ in the $+(\overrightarrow{r}_u \times \overrightarrow{r}_u)$ direction, so use $+\iint$.
- If $\mathcal{S}$ is closed (so no boundaries) and $\overrightarrow{n}$ is not specified, then one assumes that $\overrightarrow{n}$ is the normal vector outward.
- If $\mathcal{S}$ is a graph of some function $z = f(x,y)$, and $\overrightarrow{n}$ is not specified, then one can assume that $\overrightarrow{n}$ is overall upward, i.e. choosing $\overrightarrow{r}(u,v) = \langle u, v, f(u,v)\rangle$ will give $+\iint$.

Example 34
Find the flux of $\vec{\mathbf{F}} = \langle z, y, x\rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution We need a parametrization for $\mathcal{S}$, so we have

$$\vec{\mathbf{r}}(u,v) = \langle \cos u \sin v, \sin u \sin v, \cos v\rangle$$

$$\vec{\mathbf{r}}_u = \langle -\sin u \sin v, \cos u \sin v, 0\rangle$$

$$\vec{\mathbf{r}}_v = \langle \cos u \cos v, \sin u \cos v, -\sin v\rangle$$

Thus

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v\rangle.$$

Note that $\mathcal{S}$ is a sphere that is closed and the orientation of the normal vector is not specified, so we assume that the normal vector is outward. To check if $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is in the direction of the normal vector or the opposite, we check the value of $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ at easy (u,v) coordinates. At $(u,v) = (0, \pi/2)$, we have

$$\vec{\mathbf{r}}(0, \pi/2) = \langle 1, 0, 0\rangle$$

$$\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \langle -1, 0, 0\rangle$$

Thus at $(1,0,0)$, since $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is going inward to the center, it is in the opposite direction of the normal vector. We have

$$D = \{0 \le u \le 2\pi, 0 \le v \le \pi\}$$

and hence

$$\begin{aligned}\iint_{\mathcal{S}} \langle z, y, x \rangle \, d\vec{\mathcal{S}} &= -\iint_D \vec{\mathbf{F}}\left(\vec{\mathbf{r}}(u,v)\right) \cdot \left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right) \, dA \\ &= -\int_{v=0}^{\pi} \int_{u=0}^{2\pi} \left(\sin^2 u \sin^3 v + 2\cos u \sin^2 v \cos u\right) \, dudv \\ &= \frac{4}{3}\pi.\end{aligned}$$

Note that at most times, no matter which point you pick on the surface, it will give a consistent direction (so all of them are going to be in the direction of the normal vector or all of them are going to be the opposite direction). If you get a complicated parametrization that there are some two points on the surface that the orientation is different, one should split the surface integral into two surface integrals. To split the surface into two pieces, one should determine the orientation before splitting to compute the integral.

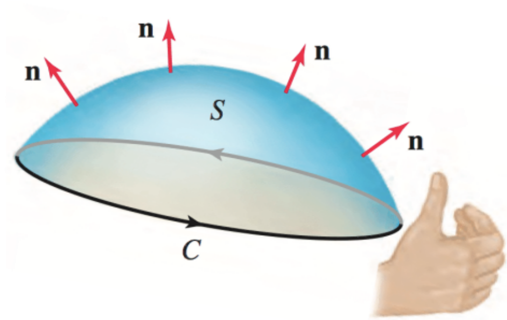
5.8 Stokes' Theorem

Theorem 5.5: Stokes

$$\iint_{\mathcal{S}} \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathcal{S}} = \oint_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

The theorem talks about the relationship between a surface integral and line integral. How does the orientation of $\mathcal{S}$ and $\partial\mathcal{S}$ relate?

The orientation of $\partial\mathcal{S}$ so that curling the right in the positive $\partial\mathcal{S}$ direction gives that the thumb is the direction of the normal vector for $\mathcal{S}$.



An alternative way to think is if the normal vector is a person walking upright on $\mathcal{S}$ towards $\partial\mathcal{S}$, then they should turn left once they arrive $\partial\mathcal{S}$. Then $\mathcal{S}$ will be on their left.

Remark.

If you start with a line integral, you have the freedom to choose any surface such that boundary is the given curve.

Example 35

Evaluate $\int_{\mathcal{C}} \langle -y^2, x, z^2 \rangle \cdot d\overrightarrow{r}$ where $\mathcal{C}$ is the intersection of the plane $y + z = 2$ and cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above.

Solution We want $\mathcal{C}$ be the boundary of some surface $\mathcal{S}$, we have a choice to choose such $\mathcal{S}$. We use the region bounded by $\mathcal{C}$ on the plane $y + z = 2$. The normal vector of $\mathcal{S}$ points in the direction of $\langle 0, 1, 1 \rangle$, so

$$\begin{aligned}\oint_{\mathcal{C}} \langle -y^2, x, z^2 \rangle \cdot d\overrightarrow{r} &= \iint_{\mathcal{S}} \operatorname{curl}\langle -y^2, x, z^2 \rangle \cdot d\overrightarrow{\mathcal{S}} \\ &= \iint_{\mathcal{S}} \langle 0, 0, 1 + 2y \rangle \cdot d\overrightarrow{\mathcal{S}}.\end{aligned}$$

We use the parametrization $\overrightarrow{r}(u, v)$ since the surface is on the plane $y + z = 2$. This gives

$$\begin{aligned}\overrightarrow{r}_u &= \langle 1, 0, 0 \rangle \\ \overrightarrow{r}_v &= \langle 0, 1, -1 \rangle \\ \overrightarrow{r}_u \times \overrightarrow{r}_v &= \langle 0, 1, 1 \rangle.\end{aligned}$$

The domain is the unit circle, $D = \{(u, v) \mid u^2 + v^2 = 1\}$. Thus

$$\begin{aligned}\iint_{\mathcal{S}} \langle 0, 0, 1 + 2y \rangle \cdot d\overrightarrow{\mathcal{S}} &= + \iint_{D} \langle 0, 0, 1 + 2v \rangle \cdot \langle 0, 1, 1 \rangle \; dA \\ &= \iint_{D} (1 + 2v) \; dA \\ &= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r(1 + 2r \sin\theta) \; d\theta dr \\ &= \pi.\end{aligned}$$

Example 36

Find $\iint_{\mathcal{S}} \operatorname{curl}\langle xz, yz, xy \rangle \cdot d\overrightarrow{\mathcal{S}}$ where $\mathcal{S}$ is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above xy-plane within the cylinder $x^2 + y^2 = 1$, oriented downwards.

Solution The boundary of $\mathcal{S}$ is

$$\partial\mathcal{S} = \{(x, y, z) \mid x^2 + y^2 = 1, z = \sqrt{3}\}$$

Since $\mathcal{S}$ is oriented downwards, $\partial\mathcal{S}$ should be oriented clockwise. We use the parametrization $\overrightarrow{r}(t) = \langle \sin t, \cos t, \sqrt{3}\rangle$. Thus

$$\oint_{t=0}^{2\pi} \langle \sqrt{3}\sin t, \sqrt{3}\cos t, \sin t\cos t\rangle \cdot \langle \cos t, -\sin t, 0\rangle = 0.$$

We get the same result with the parametrization $\overrightarrow{r}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle$ with a minus sign since this parametrization is counterclockwise.

Remark.

Stokes' theorem is saying that $\iint_{\mathcal{S}_1} \operatorname{curl} \overrightarrow{\mathbf{F}} = \iint_{\mathcal{S}_2} \overrightarrow{\mathbf{F}}$ as long as $\partial S_1 = \partial S_2$.

5.9 Divergence Theorem

Theorem 5.6: Divergence Theorem

$\iiint_E \operatorname{div} \overrightarrow{\mathbf{F}} = \iint_{\partial E} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathcal{S}}$ where ∂E is oriented outward.

Note that we can use the term *outward* since E will be covered completed by ∂E.

Example 37

If E is a solid sphere, then ∂E is the sphere surface, and if E is a solid cube, then ∂E is six faces.

Example 38

Find the flux of $\overrightarrow{\mathbf{F}} = \langle z, y, x\rangle$ across the unit sphere $\mathcal{S}$.

Solution Let E be the solid unit sphere. The flux is

$$\begin{aligned}\iint_S \langle z, y, x\rangle \cdot d\overrightarrow{S} &= \iiint_E \operatorname{div}\langle z, y, x\rangle \; dV \\ &= \iiint_E 1 \; dV \\ &= \frac{4}{3}\pi.\end{aligned}$$