Numerical Solutions to Laplace's Equation

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Applied Mathematics Project

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Differential Equations

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There are two types of differential equations, namely *Ordinary Differential Equations* and *Partial Differential Equations*. Ordinary Differential Equations are equations containing only the normal derivative, and Partial Differential Equations are equations containing partial derivatives.

Examples

An example of an ordinary differential equation is the *Bessel's Equation*

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

and an example of a partial differential equation is the *Heat Equation*

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

where u is a function of two variables x and t.

Laplace's Equation

The **Laplace's Equation** is a differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

with the *boundary conditions*

$$u(0, y) = f_1(y),\ u(a, y) = f_2(y),\ u(x, 0) = g_1(x),\ u(x, b) = g_2(x).$$

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The region R is the rectangular region $[0, a] \times [0, b]$ for now, but it can be any simply connected region.

Product Solutions to the Laplace's Equation

Partial differential equations, in general, don't have an explicit formula. However, if we assume that the solution $u(x, y)$ is *separable*, i.e. there exists a function $X(x)$ and $Y(y)$ such that

$$u(x, y) = X(x)Y(y),$$

then we can find a solution in a rectangular region R.

Product Solutions to the Laplace's Equation

The solution to the Laplace's equation on a rectangular region R is

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where

$$u_1(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x \text{ and}$$

$$u_2(x, y) = \sum_{n=1}^{\infty} \left(C_n \cosh \frac{n\pi}{b} x + D_n \sinh \frac{n\pi}{b} x \right) \sin \frac{n\pi}{b} y$$

Product Solutions to the Laplace's Equation

Here, the coefficients A_n, B_n, C_n, and D_n are

$$A_n = \frac{2}{a}\int_0^a g_1(x)\sin\frac{n\pi}{a}x\,dx$$

$$B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g_2(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$$

$$C_n = \frac{2}{b}\int_0^b f_1(y)\sin\frac{n\pi}{b}y\,dy$$

$$D_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b f_2(y)\sin\frac{n\pi}{b}y\,dy - C_n\cosh\frac{n\pi}{b}a\right).$$

Numerical Methods

Even though it's hard to find an explicit formula, we can try some numerical methods. Numerical methods are methods of finding values of $u(x_1, y_1)$ for points (x_1, y_1) of the solution function. Here. we use the *five-point approximation method*.

The Five-point Approximation Method

We start with the identity

$$f''(x) = \lim_{h \to 0} \frac{1}{h^2} \big(f(x+h) - 2f(x) + f(x-h) \big).$$

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Proof.

$$\lim_{h\to 0} \frac{1}{h^2}\big(f(x+h) - 2f(x) + f(x-h)\big) = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

$$= \frac{1}{2}\left(\frac{f'(x+h) - f'(x)}{h} - \frac{f'(x-h) - f'(x)}{h}\right) = f(x).$$

Then, we have

$$f''(x) \approx \lim_{h\to 0} \frac{1}{h^2}\big(f(x+h) - 2f(x) + f(x-h)\big)$$

when h is small enough.

The Five-point Approximation Method

Generalizing to functions of two variables, we have

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{h^2}\big(u(x+h,y) - 2u(x,y) + u(x-h,y)\big) \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{1}{h^2}\big(u(x,y+h) - 2u(x,y) + u(x,y-h)\big).$$

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$$\frac{\partial^2 u}{\partial y^2} \approx \frac{1}{h^2}\big(u(x,y+h) - 2u(x,y) + u(x,y-h)\big).$$

Then, the equation becomes

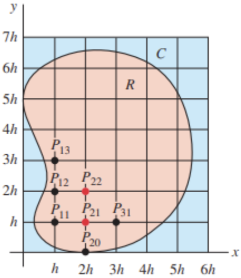
$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) = 0.$$

Mesh Size and Mesh Points

The equation now is

$$u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)=0.$$

For a simply-connected region R, consider a rectangular grid consisting of horizontal lines spaced h units apart and vertical lines spaced h units apart placed over R. Here, h is called the *mesh size*, and the points are called the *mesh points*.



Mesh Size and Mesh Points

If we let $u(x, y) = u_{ij}$ where u_{ij} is the intersection with the ith vertical line and jth horizontal line, then we have

$$u(x+h, y) = u_{i+1,j},\ u(x-h, y) = u_{i-1,j} \text{ and}$$
$$u(x, y+h) = u_{i,j+1},\ u(x, y-h) = u_{i,j-1}.$$

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The equation becomes

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0,$$

and therefore we have

$$u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}),$$

which shows that u_{ij} is the average value of u at the four neighboring points.

Example

Suppose that we need to solve the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

on a rectangular region $[0,2] \times [0,2]$ with the boundary conditions

$$u(0,y) = 0,\ u(2,y) = -y^2 + 2,\ 0 < y < 2$$

$$u(x,0) = 0,\ u(x,2) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}.$$

Example - Solution

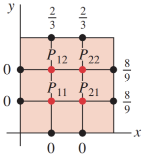
We let the mesh size $h = 2/3$. Then, we need to solve for u_{11}, u_{12}, u_{21}, u_{22} where

$$u_{11} = u\left(\frac{2}{3}, \frac{2}{3}\right),\ u_{12} = u\left(\frac{2}{3}, \frac{4}{3}\right),\ u_{21} = u\left(\frac{4}{3}, \frac{2}{3}\right), \text{ and } u_{22} = u\left(\frac{4}{3}, \frac{4}{3}\right).$$

The boundary conditions give that

$$u_{10} = u_{20} = 0,\ u_{13} = u_{23} = \frac{2}{3}$$

$$u_{01} = u_{02} = 0,\ u_{31} = u_{32} = \frac{8}{9}.$$



Example - Solution

Then, by the five-point approximation formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0,$$

we have

$$u_{21} + u_{12} + u_{01} + u_{10} - 4u_{11} = 0$$

when $i = 1$ and $j = 1$. With the boundary conditions, the equation becomes

$$-4u_{11} + u_{21} + u_{12} = 0.$$

Example - Solution

Using the formula for other mesh points, we get a system of linear equations

$$\begin{aligned}
-4u_{11} + u_{21} + u_{12} \qquad\qquad &= 0 \\
u_{11} - 4u_{21} \qquad + u_{22} &= -\frac{8}{9} \\
u_{11} \qquad - 4u_{12} + u_{22} &= -\frac{2}{3} \\
+ u_{21} + u_{12} - 4u_{22} &= -\frac{14}{9}.
\end{aligned}$$

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+ u_{21} + u_{12} - 4u_{22} &= -\frac{14}{9}.
\end{aligned}$$

Solving for these gives

$$u_{11} = 0.1944,\ u_{21} = 0.4167,\ u_{12} = 0.3611, \text{ and } u_{22} = 0.5833.$$

Matrix Form

Notice that the equation above can be written as

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{8}{9} \\ -\frac{2}{3} \\ -\frac{14}{9} \end{pmatrix}.$$

Higher Accuracy Example

Suppose that we need to solve the Laplace's equation

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on a rectangular region $[0,2] \times [0,2]$ with the boundary conditions

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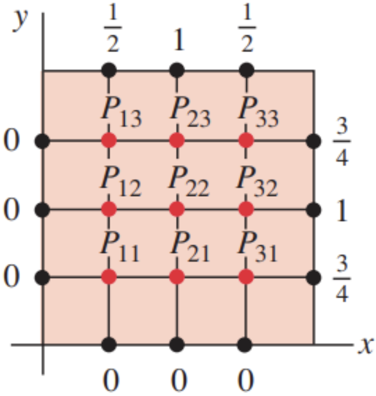
$$u(x,0) = 0,\ u(x,2) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

with higher accuracy.

Higher Accuracy Example

We let the mesh size $h = 1/2$. Then, we now have 9 variables and 9 equations instead of 4.



Higher Accuracy Example

$$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{4} \\ 0 \\ 0 \\ -1 \\ -\frac{1}{2} \\ -1 \\ -\frac{5}{4} \end{pmatrix},$$

Higher Accuracy Example

$$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{4} \\ 0 \\ 0 \\ -1 \\ -\frac{1}{2} \\ -1 \\ -\frac{5}{4} \end{pmatrix},$$

and we have

$$u_{11} = 0.1094,\ u_{21} = 0.2277,\ u_{31} = 0.3951$$

$$u_{12} = 0.2098,\ u_{22} = 0.4063,\ u_{32} = 0.6027$$

$$u_{13} = 0.3237,\ u_{23} = 0.5848,\ u_{33} = 0.6094.$$

Accuracy of the Approximation

With mesh size of $h = 2/3$, we got 4 mesh points and a linear system of four equations, and with mesh size $h = 1/2$, we got 9 mesh points and equations. With smaller mesh size, we will get more mesh points and therefore the approximation will become more accurate. However, more mesh points mean that there are more equations to solve. In a rectangular region $[0, a] \times [0, b]$, if we set $h = a/m = b/n$ for some natural numbers m and n, then there are $(m-1)(n-1)$ points, and $(m-1)(n-1)$ equations to solve. To gain high accuracy, it takes a long time to solve the system by hand, so we use MATLAB.

Code - Setup

```
% Setup
a = 2.0;  % x-dimension
b = 5.0;  % y-dimension
h = 0.05; % mesh size

% Boundary conditions
f1 = @(y) 0;                % u(0, y) = f1(y)
f2 = @(y) y^2;            % u(a, y) = f2(y)
g1 = @(x) 0;                % u(x, 0) = g1(x)
g2 = @(x) x^3;            % u(x, b) = g2(x)

% Setup done

Nx = a/h - 1; % Number of grid points in x-direction
Ny = b/h - 1; % Number of grid points in y-direction
siz = Nx*Ny;

% Initialize the grid
x = linspace(0, a, Nx+2);
y = linspace(0, b, Ny+2);

% Initialize the matrix
u = zeros(siz, siz);
```

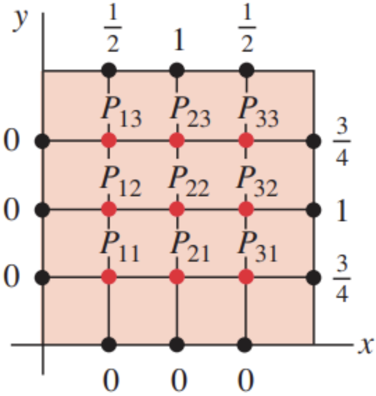
Code - Sparse Matrix

```
% Matrix
    for i = 2:Nx-1
        for j = 2:Ny-1
            u(i+Nx*j-Nx, i+Nx*j-Nx) = -4;
u(i+Nx*j-Nx, i+Nx*j-Nx-1) = 1;
u(i+Nx*j-Nx, i+Nx*j-Nx+1) = 1;
u(i+Nx*j-Nx, i+Nx*j) = 1;
u(i+Nx*j-Nx, i+Nx*j-2*Nx) = 1;
        end
    end

% all diagonal are -4
for i = 1:siz
u(i,i)=-4;
end

% u(1,1)
u(1,2) = 1;
u(1,1+Nx) = 1;

% u(1,Ny)
u(siz-Nx+1,siz-Nx+2)=1;
u(siz-Nx+1,siz-2*Nx+1)=1;

% u(Nx,1)
u(Nx,Nx-1)=1;
u(Nx,2*Nx)=1;

% u(Nx,Ny)
u(siz, siz-1) = 1;
u(siz, siz-Nx) = 1;
```

```
% u(x,1)
for i=2:Nx-1
u(i,i-1)=1;
u(i,i+1)=1;
u(i,i+Nx)=1;
end

% u(x,Ny)
for i=2:Nx-1
u(siz-Nx+i,siz-Nx+i-1)=1;
u(siz-Nx+i,siz-Nx+i+1)=1;
u(siz-Nx+i,siz-2*Nx+i)=1;
end

% u(1,y)
for i=2:Ny-1
u(i*Nx-Nx+1,i*Nx-2*Nx+1)=1;
u(i*Nx-Nx+1,i*Nx+1)=1;
u(i*Nx-Nx+1,i*Nx-Nx+2)=1;
end

% u(Nx,y)
for i=2:Ny-1
u(i*Nx,i*Nx-Nx)=1;
u(i*Nx,i*Nx+Nx)=1;
u(i*Nx,i*Nx-1)=1;
end
```

Classification of Mesh Points

To make the sparse matrix and the constant matrix, mesh points are classified into points in the *center*, *boundary*, and *corner*.



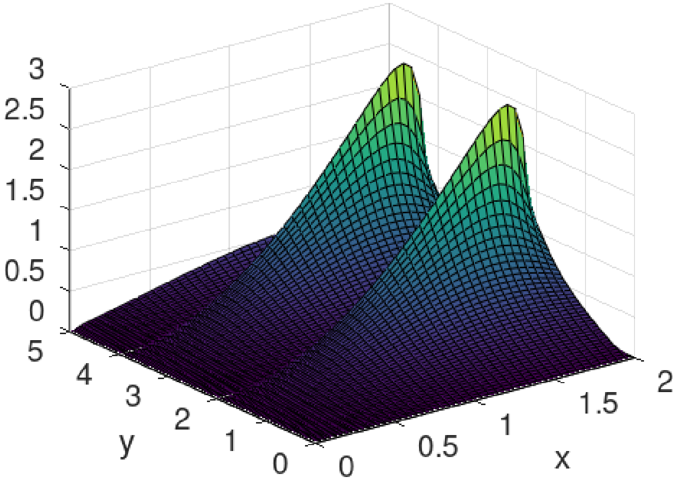
Code - Constant and Solution Matrix

```
% Constant matrix
c(1,1) = -1*f1(h)-g1(h);
c(siz-Nx+1,1) = -1*f1(b-h)-g2(h);
c(Nx,1) = -1*f2(h)-g1(a-h);
c(siz,1) = -1*f2(b-h)-g2(a-h);
for i=2:Nx-1
c(i,1) = -1*g1(i*h);
end
for i=2:Nx-1
c(siz-Nx+i,1) = -1*g2(i*h);
end
for i=2:Nx-1
c(i*Nx-Nx+1,1) = -1*f1(i*h);
end
for i=2:Nx-1
c(i*Nx,1) = -1*f2(i*h);
end

s = linsolve(u,c);
```

Code - Visualization

```
% Plot the solution
z = zeros(Ny+2,Nx+2);
for i=1:Ny
for j=1:Nx
z(i+1,j+1) = s(j*Nx-Nx+i);
end
end
surf(x,y,z)
xlabel('x');
ylabel('y');
zlabel('u');
```

Code - Results



Thanks for listening!