PROMYS - Modular Forms

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Modular Forms is a course that I took in summer 2023, in the summer program PROMYS (Program in Mathematics for Young Scientists) by Prof. David Rohrlich. Sections are divided by each week, and subsections are divided by each day of lecture. Problem sets were given every Monday, but listed in the last subsection of a section. I thank Diana Harambas, Emmy Huang, Eamon Zhang, and Vincent Tran for helping me with taking notes.

1 Week 1

1.1 July 3, 2023

Definition 1.1: $SL_2(\mathbb{Z})$

The modular group $SL_2(\mathbb{Z})$ is the set of 2×2 matrices with integer entries such that their determinant is 1, under matrix multiplication. In other words, if is the set of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $det(\gamma) = ad - bc = 1$.

Example 1

$\gamma = \begin{pmatrix} 7 & 17 \\ 2 & 5 \end{pmatrix} \in SL_2(\mathbb{Z})$ since $det(\gamma) = 7 \cdot 5 - 17 \cdot 2 = 1$.

Definition 1.2: Matrix Multiplication

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then the **matrix multiplication** is defined by

$$\gamma\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd'. \end{pmatrix}$$

Remark. (Matrix multiplication is not commutative)

Consider $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

so $TS \neq ST$.

Theorem 1.1: $SL_2(\mathbb{Z})$ is a group

$\mathrm{SL}_2(\mathbb{Z})$, the modular group has the following properties:

- Associativity: For any $\gamma, \gamma', \gamma''$, $(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma'')$
- Identity element: There exist an identity element $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ such that
$$\gamma I = I\gamma = \gamma$$
- Inverse element: For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there exist an inverse element $\gamma' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{Z})$ such that
$$\gamma\gamma' = \gamma'\gamma = I$$

Definition 1.3: Fractional Linear Transformation

Define $H = \{x+yi \mid y > 0\}$. Then the group $SL_2(\mathbb{Z})$ acts on H by **fractional linear transformations**. That is, given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $z \in H$,

$$\gamma z = \frac{az+b}{cz+d}.$$

Claim. $\Im(\gamma z) > 0$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and $z = x + yi \in H$. Then

$$\begin{aligned}\gamma z = \frac{az+b}{cz+d} &= \frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d} \\ &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{\big((ax+b)+i(ay)\big)\big((cx+d)-i(cy)\big)}{|cz+d|^2}.\end{aligned}$$

Hence $\Im(\gamma z) = \dfrac{(ad-bc)y}{|cz+d|^2} = \dfrac{y}{|cz+d|^2} > 0$. ■

1.2 July 5, 2023

Define $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Example 2 (Action on T and S on H)

$$Tz = \frac{z+1}{1} = z+1$$
$$Sz = \frac{0z-1}{1z+0} = -\frac{1}{|z|} = -\frac{\bar{z}}{|z|^2}$$

In H, Tz has the effect of shifting 1 unit to the right. Also, the three points $(0,0)$, Sz, and $-\bar{z}$ is collinear.

Theorem 1.2: Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta$.

Proof. In you use the Taylor series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Then, $e^{i\theta} = 1 + i\theta - \dfrac{\theta^2}{2!} - i\dfrac{\theta^3}{3!} + \cdots = \cos\theta + i\sin\theta$. ■

Definition 1.4: Modular Form of Weight k

Fix $k \in \mathbb{Z}$. A **modular form of weight** k for $SL_2(\mathbb{Z})$ is a function $f : H \to \mathbb{C}$ such that

1. $f(\gamma z) = (cz + d)^k f(z)$
2. There exists a convergent series representation $f(z) = \sum_{n \geq 0} a(n)e^{2\pi i n z}$ on H, for some $a(1), a(2), \cdots \in \mathbb{C}$.

Example 3

Take $k = 0$, and choose any constant in $\mathbb{C}$. Say 17. Define $f(z) = 17$ for all z. Then

1. $f(\gamma z) = 17 = (cz + d)^0 f(z)$
2. $f(z) = 17 = \sum a(n)e^{2\pi i n z}$ with $a(0) = 17$ and $a(n) = 0$ for $n > 0$.

Example 4 (Non-example)

Take $k = -2$, and define $f(z) = y$. Then

$$f(\gamma z) = y(\gamma z) = \frac{y}{|cz + d|^2} = |cz + d|^{-2} f(x)$$

This is not an example of a modular form because there is an absolute value sign.

1.3 July 6, 2023

Example 5

Fix $k \in \mathbb{Z}$, $k \geq 4$ and even. Consider

$$\begin{aligned} f(z) &= \sum_{(m,n) \neq (0,0)} (mz + n)^{-k} \\ &= \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^k} \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^k} \qquad (m, n) \neq (0, 0) \end{aligned}$$

and assume convergence.

We first verify property 1.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d. \end{pmatrix}$

$$\begin{aligned} f(\gamma z) &= \sum \left(m\frac{az+b}{cz+d} + n \right)^{-k} \\ &= \frac{1}{\sum \left(m\frac{az+b}{cz+d} + n \right)^{k}} \\ &= \frac{1}{\frac{1}{(cz+d)^k}\big(m(az+b)+n(cz+d)\big)^k} \\ &= (cz+d)^k \sum_{(m,n)\neq(0,0)} \frac{1}{\big((ma+nc)z+(mb+nd)\big)^k} \\ &= (cz+d)^k \sum_{(m,n)\neq(0,0)} \frac{1}{(m'z+n')^k} \end{aligned}$$

for $m' = ma + nc$ and $n' = mb + nd$. Then, we must check that every $(m', n') \in \mathbb{Z}^2 \setminus \{(0,0\}$ occurs exactly once. In other words, given (m', n'), there is a unique solution $(m, n) \in \mathbb{Z}^2 \setminus \{(0,0\}$ to

$$\begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} m' & n' \end{pmatrix}.$$

Since $SL_2(\mathbb{Z})$ is a group, for any $\gamma \in SL_2(\mathbb{Z})$, there exists an inverse element γ^{-1}. Then $(m, n) = (m', n')\gamma^{-1}$ satisfies the equation since

$$\big((m', n')\gamma^{-1}\big)\gamma = (m', n')I = (m', n').$$

■

Therefore, property 1 holds. Also, from now on, use the notation

$$f(z) = \sum_{(m,n)\neq(0,0)} (mz+n)^{-k} = S_k(z).$$

Question. Why did we define $S_k(z)$ for only even k?

What happens when k becomes odd?

Claim. $S_k(z) = 0$ when k is odd.

Proof. Let k be odd.
Apply $f(\gamma z) = (cz+d)^k f(z)$ with $\gamma = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\gamma z = \dfrac{-z+0}{0-1} = z$, and $(cz+d)^k = (0z-1)^k = -1$. Therefore, we get $f(z) = -f(z)$, and $f(z) = 0$. ■

Lemma : Any Modular Form is Periodic

If $f(z)$ is any modular form, then $f(z+1) = f(z)$.

Proof. Apply $f(\gamma z) = (cz+d)^k f(z)$ with $\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then, since $Tz = \frac{z+1}{0z+1} = z+1$ and $(cz+d)^k = 1^k = 1$, we get

$$f(z+1) = 1^k f(z) = f(z).$$

■

To verify property 2, we must show that there is a convergent series representation

$$f(z) = \sum_{n \geq 0} a(n) e^{2\pi i n z}.$$

Problem Set 1: The modular group

Problem 1

Define the product of two 2×2 matrices by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

(In the problem and the next, we take a, b, $c\ldots$ to be real numbers.)

(a) Verify that for three 2×2 matrices γ, γ', γ'' we have $\gamma\gamma')\gamma'' = \gamma('\gamma'')$.

(b) Verify that the operation of matrix multiplication on the set of 2×2 matrices has a unique multiplicative identity, namely the matrix I with $a = d = 1$ and $b = c = 0$.

Problem 2

The *determinant* of a 2×2 matrix γ is the number $\det \gamma = ad - bc$.

(a) Verify that $\det(\gamma\gamma') = \det(\gamma)\det(\gamma')$.

(b) Show that if $\det \gamma \neq 0$ then the matrix

$$\gamma' = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(with entries $d/(ad-bc)$, $-b/(ad-bc)$, and so on) satisfies $\gamma\gamma' = 1$ and $\gamma'\gamma = 1$. Thus γ' is a multiplicative inverse of γ. Often *multiplicative* is understood, and we speak simply of an *inverse* of γ.

(c) Show that an inverse of γ, if it exists, is unique, so we can speak of *the* inverse of γ, which we henceforth denote γ^{-1} rather than γ'.

(d) In (b) we showed that the condition of $\det \gamma =\neq 0$ is *sufficient* for γ to have an inverse. Now use (a) to show that this condition is *necessary*. In other words, show that if γ has an inverse then $\det \gamma \neq 0$.

Problem 3

Henceforth we take a, b, c, ... to be *integers*, and we write $SL_2(\mathbb{Z})$ for the set of 2×2 matrices with integer coefficients and determinant 1. Show that $SL_2(\mathbb{Z})$ is closed under the operation of matrix multiplication and the operation of taking inverses, whence this operation makes $SL_2(\mathbb{Z})$ into a *group*: The operation is associative, there is a multiplicative identity, and every element has an inverse.

Problem 4

Define matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

For $n \in \mathbb{Z}$ and $\gamma \in SL_2(\mathbb{Z})$, verify that $T^n\gamma$ is the matrix obtained from γ by adding n times the second row of γ to the first row, while U^n is obtained by adding n times the first row to the second row. (In particular, T^n and U^n differ from T and U only by having an n in the upper righthand corner in the case of T and in the lower left-hand corner in the case of U.) Explain why there is a finite sequence of such "row operations" which transforms γ into the identity matrix I. (First verify that $-I = U^2T^{-1}U^2T^{-1}$.) Deduce that T and U *generate* $SL_2(\mathbb{Z})$: In other words, every element in $SL_2(\mathbb{Z})$ can be written as a finite product $T^{n_1}U^{n_2}T^{n_3}U^{n_4}\ldots$ with $n_i \in \mathbb{Z}$.

Problem 5

Illustrate the previous problem by writing the matrix

$$\gamma = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$$

explicitly as a finite product of the form $T^{n_1}U^{n_2}T^{n_3}U^{n_4}\ldots$ with $n_i \in \mathbb{Z}$.

Problem 6

Put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Verify the identity $ST^{-1}S^{-1} = U$, and deduce that S and T also generate $SL_2(\mathbb{Z})$.

Problem 7

Put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Verify that $S^{-1}R = T$ and deduce that R and S are yet another pair of generators for $SL_2(\mathbb{Z})$. It may seem silly to exhibit three different pairs of generators for $SL_2(\mathbb{Z})$, but each has its special properties. For example, show that $S^2 = R^3 = -I$.

Problem 8

Given an integer $N \geq 1$, we write $\Gamma_1(N)$ for the subset of $SL_2(\mathbb{Z})$ consisting of those γ for which $c \equiv 0$ and $a \equiv d \equiv 1$ modulo N. (Thus if $N = 1$ then $\Gamma_1(N)$ is just $SL_2(\mathbb{Z})$ itself.) Show that $\Gamma_1(N)$ is a *subgroup* of $SL_2(\mathbb{Z})$: in other words, it is closed under multiplication, it contains I, and it contains the inverse of each of its elements.

Problem 9

Let $N = 1$, 2, 3, or 4, and define T and U as in Problem 4. Put $V = U^N$. This problem is a generalization of Problem 4: The matrices T and V generate $\Gamma_1(N)$. To prove this, let Γ be the subgroup of $\Gamma_1(N)$ consisting of all finite products of the form $T^{n_1}V^{n_2}T^{n_3}V^{n_4}\ldots$ with $n \in \mathbb{Z}$. (Why is this a subgroup?) We must show that $\Gamma = \Gamma_1(N)$. Let γ denote an arbitrary element of $\Gamma_1(N)$.

(a) Show that if $a = 0$ or $c = 0$ then $\gamma \in \Gamma$. (Hint: Observe that if $a = 0$ then $N = 1$ and if $c = 0$ then $N = 1$ or 2 and in addition, $\gamma = \pm T^n$ for some n. In these cases, you can simply use Problem 4, including the identity $-I = U^2T^{-1}U^2T^{-1}$, which if $N = 1$ can be written $-I = VT^{-1}VT^{-1}$.)

(b) Use double induction on $|c|$ and $|a|$ to show that $\gamma \in \Gamma$. For the inductive step, take $ac \neq 0$ and assume that $\gamma' \in \Gamma$ for all matrices $\gamma' \in \Gamma_1(N)$ such that either $|c'| < |c|$ or $|c'| = |c|$ and $|a'| < |a|$. If $|c| \leq N|a|/2$ show that with an appropriate choice of sign the inductive hypothesis applies to the matrix $\gamma' = T^{\pm 1}\gamma$, and if $|c| > N|a|/2$ show that the same is true with $\gamma' = V^{\pm 1}\gamma$.

Problem 10

Define a function ω from $SL_2(\mathbb{Z})$ to the integers mod 12 by the formula

$$\omega(\gamma) = (1 - c^2)(db + 3d(c-1) + c + 3) + c(d + a + 3) \mod 12.$$

The purpose of this problem is to prove that $\omega(\gamma\delta) = \omega(\gamma) + \omega(\delta)$.

(a) Show that if c and d are relatively prime integers then $c^2 - (cd)^2 + d^2 \equiv 1$ modulo 3 and modulo 4, hence modulo 12.

(b) Let T be as in Problem 4. Show that $\omega(T\gamma) = \omega(T) + \omega(\gamma)$.

(c) Let U be as in Problem 4. Show that $4\omega(U\gamma) = 4\omega(U) + 4\omega(\gamma)$. You may want to consider a division into four cases as follows: (i) $a \equiv 0 \bmod 3$, (ii) $c \equiv 0 \bmod 3$, (iii) $a \equiv c \bmod 3$, (iv) $a \equiv -c \bmod 3$. Don't forget that $ad - bc = 1$!

(d) Now show that $3\omega(U\gamma) = 3\omega(U) + 3\omega(\gamma)$. Once again, a division into cases may be helpful: (i) $a \equiv 0 \bmod 2$, (ii) $c \equiv 0 \bmod 4$, (iii) $c \equiv 2 \bmod 4$, (iv) $a \equiv c \bmod 4$, and (v) $a \equiv -c \bmod 4$.

(e) Conclude that $\omega(\gamma\delta) = \omega(\gamma) + \omega(\delta)$ for all $\gamma, \delta \in SL_2(\mathbb{Z})$.

2 Week 2

2.1 July 10, 2023

Definition 2.1: Bernoulli Numbers

Define the **Bernoulli numbers** $b_0, b_1, b_2, \cdots$ by $\dfrac{t}{e^t - 1} = \sum_{k\geq 0} b_k \dfrac{t^k}{k!}$.

We know the Taylor series for $e^x : 1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \cdots$.

Since $\dfrac{t}{e^t - 1} = \sum_{k\geq 0} b_k \dfrac{t^k}{k!}$,

$$\begin{aligned} t &= (e^t - 1)\sum_{k\geq 0} \frac{t^k}{k!} \\ &= \left(t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right)\left(b_0 + b_1 t + b_2\frac{t^2}{2} + {}_-3\frac{t^3}{6} + \cdots\right) \\ &= b_0 t + \left(\frac{b_0}{2}\right)t^2 + \left(\frac{1}{6}b_0 + \frac{1}{2}b_1 + \frac{1}{2}b_2\right)t^3 + \cdots. \end{aligned}$$

Therefore, the coefficients for t^k should be all 0 for $k > 2$. If you compute some Bernoulli numbers with comparing the coefficients, you get

- $b_0 = 1$
- $b_1 = -1/2$
- $b_2 = 1/6$
- $b_3 = 0$
- $b_4 = -1/30$
- $b_5 = 0$

Theorem 2.1

If k is odd and $k \neq 1$, then $b_k = 0$.

Proof. Since the coefficient of t in $\sum_{k\geq 0}$ is $-1/2$,

$$\frac{t}{e^t-1}+\frac{1}{2}t=\sum_{k\geq 0,\, k\neq 1} b_k\frac{t^k}{k!}$$

For the left hand side,

$$\begin{aligned}\frac{t}{e^t-1}+\frac{1}{2}t &= \frac{t+\frac{1}{2}t(e^t-1)}{e^t-1}\\ &= \frac{1}{2}t\frac{e^t+1}{e^t-1}\\ &= \frac{1}{2}t\frac{e^{t/2}+e^{-t/2}}{e^{t/2}-e^{-t/2}},\end{aligned}$$

and we get that the function in the left hand side is an even function. If we substitute $-t$ into t, we get

$$\sum_{k\geq 0,\, k\neq 1} b_k(-1)^k\frac{t^k}{k!}=\sum_{k\geq 0,\, k\neq 1} b_k\frac{t^k}{k!}$$

Therefore, for odd k that is not 1, $b_k=(-1)^k b_k=-b_k$ and $b_k=0$. ∎

Definition 2.2: Bernoulli Polynomials

Define polynomials $B_0(x)$, $B_1(x)$, $B_2(x)$, $\cdots$ by

$$\frac{te^{tx}}{e^t-1}=\left(\sum_{k\geq 0} b_k\frac{t^k}{k!}\right)\left(\sum_{j\geq 0} x^j\frac{t^j}{j!}\right)=\sum_{k\geq 0} B_k(x)\frac{t^k}{k!}.$$

If we expand the formula, we get

$$\left(\sum_{k\geq 0} b_k\frac{t^k}{k!}\right)\left(\sum_{j\geq 0} x^j\frac{t^j}{j!}\right)=B_0(x)+B_1(x)+B_2(x)+\cdots$$

Comparing the coefficients of t^k, we can get $B_k(x)$. Some Bernoulli polynomials are:

1. $B_0(x)=1$
2. $B_1(x)=x-\frac{1}{2}$
3. $B_2(x)=x^2-x+\frac{1}{6}$.

Claim. $B_k(0) = b_k$.

Proof. In the formula $\dfrac{te^{tx}}{e^t-1} = \sum_{k\geq 0} B_k(x)\dfrac{t^k}{k!}$, if we substitute $x = 0$, then we get

$$\sum_{k\geq 0} B_k(x)\frac{t^k}{k!} = \frac{te^{tx}}{e^t-1} = \sum_{k\geq 0} b_k\frac{t^k}{k!}. \qquad \blacksquare$$

Claim. $B_k(0) = B_k(1)$ for $k \neq 1$.

Proof. Assume $k \neq 1$. In the formula $\dfrac{te^{tx}}{e^t-1} = \sum_{k\geq 0} B_k(x)\dfrac{t^k}{k!}$, if we substitute $x = 1$, then we get

$$\frac{te^t}{e^t-1} = \sum_{k\geq 0} B_k(1)\frac{t^k}{k!}.$$

Then,

$$\frac{te^t}{e^t-1} - \frac{t}{e^t-1} = \sum_{k\geq 0} \left(B_k(1) - B_k(0)\right)\frac{t^k}{k!} = t.$$

Since $\dfrac{B_k(1)-B_k(0)}{k!} = 0$ for $k \neq 1$, we get $B_k(1) = B_k(0)$ for $k \neq 1$. $\blacksquare$

2.2 July 12, 2023

Define $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $S^2 = -I = R^3$.

Consider the geometric series $1 + r + r^2 + \cdots + r^n$. If $r \neq 1$, then since $(1-r)(1+r+r^2+\cdots+r^n) = 1 - r^{n+1}$, we get

$$1 + r + r^2 + \cdots = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}.$$

Since $\lim_{n\to\infty} \dfrac{r^{n+1}}{1-r} = 0$ if $0 \leq r < 1$, $1 + r + r^2 + \cdots = \dfrac{1}{1-r}$ if $0 \leq r < 1$.

Definition 2.3: Riemann-Zeta Function

The **Riemann-Zeta Function** is defined by

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

for $s > 1$.

Theorem 2.2: Convergence of the Riemann-Zeta Functions

The Riemann-Zeta Function converges for $s > 1$.

Proof.

$$\begin{aligned}\zeta(s) &= 1 + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s}\right) + \cdots \\ &\leq 1 + \left(\frac{1}{2^s} + \frac{1}{2^s}\right) + \left(\frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s}\right) + \cdots \\ &= 1 + 2^{1-s} + 4^{1-s} + \cdots \\ &= 1 + r + r^2 + \cdots\end{aligned}$$

converges if $2^{1-s} < 1$, hence $s > 1$. ■

Theorem 2.3: Divergence of the Harmonic Series

When $s = 1$, the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

Proof.

$$\begin{aligned}1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

diverges. ■

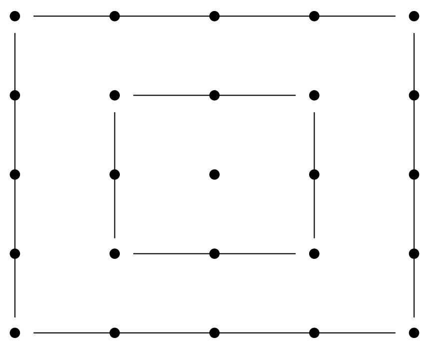
2.3 July 13, 2023

Recall that for k even and $k \geq 4$, $S_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-k}$

Theorem 2.4: Convergence of $S_k(z)$

$S_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-s}$ converges for $s > 2$.

Proof. Let $S_N = \{(m,n) \mid \max\{|m|,|n|\} = N\}$.



Consider the following diagram, where the inner border is S_1, and the outer border is S_2. We get that

$$\begin{aligned}|S_N| &= (2N+1)^2 - \big(2(N-1)+1\big)^2 \\ &= (2N+1)^2 - (2N-1)^2 = 8N\end{aligned}$$

We know, from Problem Set 2 P12, $|mz+n| \geq C \cdot \max\{|m|,|n|\} = CN$, so $|mz+n|^{-1} \leq 1/CN$. Finally,

$$\begin{aligned}S_k(z) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-s} \\ &= \sum_{N\geq 1} \sum_{(m,n)\in S_N} (mz+n)^{-s} \\ &\leq \sum_{N\geq 1} 8N \cdot \frac{1}{(CN)^s} \\ &= \frac{8}{C^s} \sum_{N\geq 1} N^{1-s}\end{aligned}$$

Because $\sum_{N\geq 1} N^{1-s}$ converges for $1-s<1$, we get $s>2$. ■

Now the goal is to find a Fourier expansion for $S_k(z)$ when k is even and $k \geq 4$. That is, we want

$$S_k(z) = \sum_{n\geq 0} a(n) e^{2\pi i n z}$$

for some $a(i)$.

Problem Set 2: Bernoulli numbers and Bernoulli polynomials

Problem 1

Show that

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} x^j b_{k-j}.$$

This formula is sometimes written as $B_k(x) = (x+b)^k$.

Problem 2

While it is not literally true that $B_k(x)$ is an even or odd function according as k is even or odd, this assertion is close to being true in at least two different ways:

(a) Show that $B_k(1-x) = (-1)^k B_k(x)$, whence the assertion is literally true for the function $f(x) = B_k(x+1/2)$. (Hint: $te^{t(1-x)}/(e^t-1) = -te^{-tx}/(e^{-t}-1)$.)

(b) Show that the assertion is also true for the function $f(x) = B_k(x) + kx^{k-1}/2$.

Problem 3

Prove that $B_k'(x) = kB_{k-1}(x)$, and use this formula together with the fact that $B_k(0) = b_k$ to compute $B_k(x)$ recursively for $k \leq 5$. Of course, we already know that $B_0(x) = 1$, $B_1(x) = x-1/2$, and $B_2(x) = x^2-x+1/6$. If you prefer, use the formula $B_k(x) = (x+b)^k$ instead of the formula $B_k'(x) = kB_{k-1}(x)$, or use some combination of the two approaches.

Problem 4

Let k and n be positive integers. Give two proofs of the identity

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{B_{k+1}(n+1-b_{k+1}}{k+1}$$

as follows: (i) set $x = n+1$ and $x = 0$ in the definition

$$\frac{te^{tx}}{(e^t-1)} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}$$

and take the difference. (ii) Integrate both sides of the definition with respect to x, say from $x = u$ to $x = u+1$, and sum from $u = 1$ to $u = n$.

Problem 5

Show that $(1+2+3+\cdots+n)^2 = 1+2^3+3^3+\cdots+n^3$.

Problem 6

Let N be a positive integer. Prove the "distribution relation"

$$N^{k-1}\sum_{j=0}^{N-1} f\big((x+j)/N\big) = f(x),$$

where $f = B_k$.

Problem 7

This problem can be viewed as a converse to Problem 6. Fix an integer $k \geq 0$, and suppose that f is a polynomial which satisfies the distribution relation in Problem 6 for every positive integer N. Prove that $f = cB_k$ for some constant c.

Problem 8

Let S and R be as in Problem 6 and 7 of Problem Set 1.

(a) Show that $Si = i$ and that $Re^{2\pi i/3} = e^{2\pi i/3}$.

(b) Let f be a modular form of weight k for $SL_2(\mathbb{Z})$. Show that if $k \not\equiv 0 \bmod 4$ then $f(i) = 0$ and that if $k \not\equiv 0 \bmod 3$ then $f(e^{2\pi i/3}) = 0$.

Problem 9

Let $GL_2^+(\mathbb{R})$ denote the group of 2×2 matrices with *real* coefficients and *positive* determinant. For $\gamma \in GL_2^+(\mathbb{R})$ and $z \in H$, we define

$$\gamma z = \frac{az+b}{cz+d}$$

as before. Check that $\gamma z \in H$, and then verify that the map $(\gamma, z) \mapsto \gamma z$ is a *left action* of the group $GL_2^+(\mathbb{R})$ on the set H: In other words, show that $Iz = z$ and $\gamma(\gamma' z) = (\gamma\gamma')z$.

Problem 10

Fix an integer $k \geq 0$. Given a complex-valued function f of H and a matrix $\gamma \in GL_2^+(\mathbb{R})$, define a function $f\,|_k\,\gamma$ of H by

$$(f\,|_k\,\gamma)(z) = \det(\gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

Since k is fixed, we can write $f\,|_k\,\gamma$ simply as $f\,|\,\gamma$. Verify that the map $(\gamma, f) \mapsto f\,|\,\gamma$ is a *right action* of $GL_2^+(\mathbb{R})$ on the set of complex-valued functions on H: In other words, $f\,|\,I = f$ and $f\,|\,(\gamma\gamma') = (f\,|\,\gamma)\,|\,\gamma$.

Problem 11

In class, we used the convergence of the geometric series $\sum_{n\geq 0} r^n$ with $r = 2^{1-s}$ to prove that the series $\sum_{n\geq 1} n^{-s}$ converges for $s > 1$. This problem gives an alternative proof. Write $s = 1 + \epsilon$, so that $\epsilon > 0$.

(a) Show that there is a positive constant c depending on ϵ such that

$$cx \leq (1+x)^\epsilon - 1$$

for $0 \leq x \leq 1$. (Using calculus, one sees that c can be chosen to be the minimum value of $\epsilon(1+x)^{\epsilon-1}$ on the interval $[0, 1]$.)

(b) By writing $n^{-\epsilon} - (n+1)^{-\epsilon} = (n+1)^{-\epsilon}\big((1+1/n)^\epsilon - 1\big)$ and applying (a) with $x = 1/n$, show that $n^{-\epsilon} - (n+1)^{-\epsilon} \geq cn^{-1}(n+1)^{-\epsilon}$. Deduce that

$$(n+1)^{-s} \leq c^{-1}\big(n^{-\epsilon} - (n+1)^{-\epsilon}\big)$$

and sum over n to complete the proof.

Problem 12

This problem leads to an inequality of the form $|mz+n| \geq CN$ for $z \in H$ and m, $n \in \mathbb{Z}$, where $N = \max(|m|, |n|)$ and C is a positive constant depending on z. Recall that an inequality of this form was used to prove the absolute convergence of the series $\sum_{(m,n)\neq(0,0)}(mz+n)^{-k}$ for $k > 2$.

(a) Prove that for u, u', v, $v' \in \mathbb{R}$ we have

$$|uu' + vv'| \leq \sqrt{u^2+v^2}\sqrt{(u')^2+(v')^2}.$$

(This is a special case of the *Cauchy-Schwarz inequality*.)

(b) Observe that $n = (mx+n) + (-x/y)(my)$, and apply (a) with $u = mx+n$, $v = my$, $u' = 1$, and $v' = -x/y$, obtaining

$$\sqrt{1+(x/y)^2}|mz+n| \geq |n|.$$

Deduce that $|mz+n| \geq CN$ with $C = \min(1/\sqrt{1+(x/y)^2}, y)$.

3 Week 3

3.1 July 17, 2023

Definition 3.1: Fractional Part Function

For any $x \in \mathbb{R}$, define the **fractional part function** $\{x\}$ by $0 \leq \{x\} < 1$, and $x = \{x\} + n$ for some $n \in \mathbb{Z}$.

Definition 3.2: Bernoulli Functions

Define **Bernoulli functions** $\mathbb{B}_k(x) = B_k(\{x\})$ where $B_k(x)$ is a Bernoulli polynomial.

For example,

- $\mathbb{B}_0(x) = 1$
- $\mathbb{B}_1(x) = \{x\} - \frac{1}{2}$
- $\mathbb{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$.

Remark.

Since $B_k(0) = B_k(1)$ for $k \neq 1$, $\mathbb{B}_k(x)$ is continuous for $k \neq 1$.

3.2 July 19, 2023

We assume that there exists a Fourier series for $\mathbb{B}_k(x)$. We now will find the Fourier series.

Lemma

For $n = 0, 1, 2, \ldots$,

$$\int_0^1 e^{2\pi inx} dx = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Proof. If $n = 0$, then $\int_0^1 e^{2\pi inx} dx = \int_0^1 1 dx = x\Big|_0^1 = 1$. If $n \neq 0$, then

$$\int_0^1 e^{2\pi inx} dx = \frac{e^{2\pi inx}}{2\pi in}\Bigg|_0^1$$

$$= \frac{1}{2\pi in}(e^{2\pi in} - 1) = 0.$$

■

Suppose $f(x) = \sum_{m \in \mathbb{Z}} a(m) e^{2\pi imx}$.

Question. How do you find $a(n)$ for a given n?

$$\begin{aligned} f(x) &= \sum_{m\in\mathbb{Z}} a(m)e^{2\pi imx} \\ f(x)e^{-2\pi inx} &= \sum_{m\in\mathbb{Z}} a(m)e^{2\pi imx-2\pi inx} \\ &= \sum_{m\in\mathbb{Z}} a(m)e^{2\pi i(m-n)x} \end{aligned}$$

If we integrate both sides from 0 to 1,

$$\begin{aligned} \int_0^1 f(x)e^{-2\pi inx}dx &= \int_0^1 \big(\sum_{m\in\mathbb{Z}} a(m)e^{2\pi i(m-n)x}\big)dx \\ &= \sum_{m\in\mathbb{Z}} a(m)\int_0^1 e^{2\pi i(m-n)x}dx \\ &= \begin{cases} \sum_{m\in\mathbb{Z}} a(m) & m=n \\ 0 & m\neq n. \end{cases} \end{aligned}$$

So $\int_0^1 f(x)e^{-2\pi inx} = a(n)$. Let's write this as $\hat{f}(n)$.

Now we will find the Fourier series of Bernoulli functions. We want $\mathbb{B}_k(x) = \sum \hat{\mathbb{B}_k}(x)e^{2\pi inx}$ for some $\hat{\mathbb{B}_k}(x)$. We use the formula that we found above.

$$\begin{aligned} \hat{\mathbb{B}_k}(x) &= \int_0^1 \mathbb{B}_k(x)e^{-2\pi inx} \qquad \text{except } k\neq 1 \text{ and } x\in\mathbb{Z} \text{ for k=1} \\ &= \int_0^1 B_k(x)e^{-2\pi inx}. \end{aligned}$$

We use the definition of Bernoulli polynomials. Since

$$\frac{te^{xt}}{e^t-1} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!,}$$

$$\begin{aligned} \int_0^1 \frac{te^{xt}}{e^t-1}e^{-2\pi inx}dx &= \sum_{k\geq 0}\left(\int_0^1 B_k(x)e^{-2\pi inx}dx\right)\frac{t^k}{k!} \\ &= \sum_{k\geq 0} \hat{\mathbb{B}_k}(x)\frac{t^k}{k!.} \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k\geq 0}\hat{\mathbb{B}_k}(x)\frac{t^k}{k!} &= \int_0^1 \frac{te^{xt}}{e^t-1}e^{-2\pi inx}dx \\ &= \frac{t}{e^t-1}\int_0^1 e^{xt}e^{-2\pi inx}dx \\ &= \frac{t}{e^t-1}\int_0^1 e^{(t-2\pi in)x}dx \\ &= \frac{t}{e^t-1}\cdot\frac{e^{t-2\pi in}-1}{t-2\pi in} \\ &= \frac{t}{e^t-1}\cdot\frac{e^t-1}{t-2\pi in} = \frac{t}{t-2\pi in}.\end{aligned}$$

If $x = 0$, we get

$$\sum_{k\geq 0}\hat{\mathbb{B}_k}(x)\frac{t^k}{k!} = \hat{\mathbb{B}_0}(0) + \hat{\mathbb{B}_1}(0)t + \hat{\mathbb{B}_2}(0)\frac{t^2}{2} + \cdots = 1,$$

so $\hat{\mathbb{B}_k}(0) = 0$ for $k \geq 1$, and $\hat{\mathbb{B}_0}(0) = 1$.

If $n \neq 0$, we get

$$\begin{aligned}\frac{t}{t-2\pi in} &= -\frac{t/2\pi in}{1-t/2\pi in} \\ &= -\sum_{k\geq 1}\left(\frac{t}{2\pi in}\right)^k = \sum_{k\geq 1}\frac{-k}{(2\pi in)^k}\cdot\frac{t^k}{k!}.\end{aligned}$$

So if $k > 1$ and $x \notin \mathbb{Z}$ if $k = 1$, $\hat{\mathbb{B}_k}(x) = -\frac{k!}{(2\pi in)^k}$ for $n \neq 0$, and

$$\begin{aligned}\mathbb{B}_k(x) &= \sum_{n\neq 0}\hat{\mathbb{B}_k}(n)e^{2\pi inx} \\ &= \sum_{n\neq 0}\frac{-k!}{(2\pi in)^k}e^{2\pi inx} \\ &= -\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}\frac{1}{n^k}e^{2\pi inx}.\end{aligned}$$

3.3 July 20, 2023

I thank Emmy Huang for helping me with notes for this session.

If we substitute $x = 0$ in the formula of $\mathbb{B}_k(x)$, since $\mathbb{B}_k(0) = B_k(0) = b_k$,

$$b_k = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{1}{n^k}.$$

Suppose $k \geq 2$, and k even. We have $\displaystyle\sum_{n \neq 0} \frac{1}{n^k} = 2 \sum_{n \in \mathbb{N}} n^{-k} = 2\zeta(k)$. So

$$b_k = -\frac{2 \cdot k!}{(2\pi i)^k} \zeta(k), \text{ and } \zeta(k) = -\frac{(2\pi i)^k b_k}{2 \cdot k!}.$$

Recall that $S_k(z) = \displaystyle\sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} (mz+n)^{-k}$. Let this be $\mathrm{I} + \mathrm{II}$ where

$$\begin{aligned}
\mathrm{I} &= \sum_{\substack{m=0 \\ n \neq 0}} (mz+n)^{-k} \\
&= \sum_{n \neq 0} n^{-k} \\
&= 2\zeta(k) = -\frac{(2\pi i)^k b_k}{k!} \\
\text{and II } &= \sum_{\substack{m \neq 0 \\ n \in \mathbb{Z}}} (mz+n)^{-k} \\
&= \sum_{m \neq 0} \big(\sum_{n \in \mathbb{Z}} (mz+n)^{-k} \big) \\
&= \sum_{m \geq 1} \big(\sum_{n \in \mathbb{Z}} (mz+n)^{-k} \big) \qquad \big((-mz-n)^k = (mz+n)^k\big)
\end{aligned}$$

If $w \in H$, $\displaystyle\sum_{n \in \mathbb{Z}} (w+n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d \geq 1} d^{k-1} e^{2\pi i d w}$, from Problem 11 of Problem Set 4.

Substitute $w = mz$ into II gives

$$\begin{aligned}
\text{II} &= 2\sum_{m\geq 1}\frac{(-2\pi i)^k}{(k-1)!}\sum_{d\geq 1} d^{k-1}e^{2\pi i dmz} \\
&= 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m\geq 1}\sum_{d\geq 1} d^{k-1}e^{2\pi i dmz} && (k \text{ is even}) \\
&= 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\Big(\sum_{d|n} d^{k-1}\Big)e^{2\pi i nz} \\
&= \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi i nz}
\end{aligned}$$

Therefore,

$$S_k(z) = -\frac{(2\pi i)^k b_k}{k!} + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi i nz}.$$

Definition 3.3: $E_k(z)$

$E_k(z)$, normalization of $S_k(z)$ is defined as

$$\begin{aligned}
E_k(z) &= -\frac{k!}{(2\pi i)^k b_k}S_k(z) \\
&= 1 - \frac{2k}{b_k}\sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi i nz}.
\end{aligned}$$

If we let $q = e^{2\pi i z}$, then

$$E_k = 1 - \frac{2k}{b_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Example 6

$$E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n = 1 + 240(q + 9q^2 + 28q^3 + \cdots)$$

Example 7

$$E_6 = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n = 1 - 504(q + 33q^2 + \cdots)$$

Example 8

Even though it is defined for $k \geq 4$, we try $k = 2$.

$$E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n.$$

However, we get a contradiction because $E(Sz) \neq (1z+0)^2 E(z)$. Alternatively, we try defining $S_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-2}$. Then,

$$\begin{aligned} S_2(\gamma z) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\gamma z + n)^2} \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{\left(m\dfrac{az+b}{cz+d} + n\right)^2} \\ &= (cz+d)^2 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{m'z+n'} \end{aligned}$$

where $\begin{pmatrix} m' & n' \end{pmatrix} = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Because this don't absolutely converge, $S_2(\gamma z) - (cz+d)^2 S_2(z) = \dfrac{12z}{2\pi i}$.

Problem Set 3: Diagonal Quadratic Forms

Problem 1

By a *positive-definite diagonal quaternary quadratic form* over $\mathbb{Z}$ we mean a polynomial of the form $f(x_1, x_2, x_3, x_4) = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2$ with integers a, b, c, $d > 0$. We denote this expression $[a, b, c, d]$. After permuting the variables x_i we may assume that f is *normalized* in the sense that $a \leq b \leq c \leq d$. We say that f is *universal* if it represents every positive integer, i.e. if for every positive integer n there is a choice of integers n_1, n_2, n_3, n_4 such that $n = f(n_1, n_2, n_3, n_4)$. In 1916 Ramanujan claimed to give a complete list of universal normalized positive-definite diagonal quaternary quadratic forms:

(a) $[1, 1, 1, i]$ with $1 \leq i \leq 7$,
(b) $[1, 1, 2, i]$ with $2 \leq i \leq 14$,
(c) $[1, 1, 3, i]$ with $3 \leq i \leq 6$,
(d) $[1, 2, 2, i]$ with $2 \leq i \leq 7$,
(e) $[1, 2, 3, i]$ with $3 \leq i \leq 10$,

(f) $[1, 2, 4, i]$ with $4 \le i \le 14$,
(g) $[1, 2, 5, i]$ with $5 \le i \le 10$.

Ramanujan's list was later proven to be correct except that the form $[1, 2, 5, 5]$ had to be removed. Why did $[1, 2, 5, 5]$ had to be removed?

Problem 2

The case $i = 1$ of (a) in Problem 1 is Lagrange's Four-Squares Theorem: Every positive integer is a sum of four squares of integers. Let

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} = 1 + \sum_{n\ge1} e^{2\pi i n^2 z}.$$

We often write $\vartheta(z) = 1 + 2\sum_{n\ge1} q^{n^2}$ with $q = e^{2\pi i z}$. Let t be a positive integer. Explain why

$$\vartheta^t = \sum_{n\ge0} r_t(n) q^n,$$

where $r_t(n)$ is the number of t-tuples of integers $(n_1, n_2, \ldots, n_t) \in \mathbb{Z}^t$ such that $n_1^2 + n_2^2 + \cdots + n_t^2 = n$. Thus, to prove the Four-Squares Theorem, it suffices to show that $r_4(n) \ge 1$ for all $n \ge 0$.

Problem 3

By the end of the program, we hope to have some idea of why modular forms can be used to prove Jacobi's formula:

$$r_4(n) = 8 \sum_{d|4, 4\nmid d} d,$$

where d runs over all positive divisors of n which are not congruent to 0 mod 4.

(a) Why does this formula prove that $r_4(n) \ge 1$ (actually $r_4(n) \ge 8$) for all n?

(b) Using Jacobi's formula, show that $r(2^\nu) = 24$ for every integer $\nu \ge 1$. Then write down the 24 elements of $\{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 : n_1^2 + n_2^2 + n_3^2 + n_4^2 = 2^\nu\}$ explicitly. You may also have to distinguish between the cases ν of even and odd.

Problem 4

Also use Jacobi's formula to show that $r_4(mn) = r_4(m)r_4(n)/8$ for coprime integers m, $n \ge 1$.

Problem 5

In class, we deduced the formula $\zeta(k) = -(2\pi i)^k b_k/(2 \cdot k!)$ (valid for k even and $k \geq 2$) by taking $x = 0$ in the Fourier expansion

$$\mathbb{B}_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n^k}$$

(valid for $k \geq 2$ or $k = 1$ and $x \in \mathbb{Z}$).

(a) By making different choices of x you can obtain other formulas. For example, show that $\sum_{n \geq 1} (-1)^{n+1} n^{-2} = \pi^2/12$.

(b) Alternatively, show that $\sum_{n \geq 1} (-1)^{n+1} n^{-s} = (1 - 2^{1-s})\zeta(s)$ for $s > 1$, and thus derive the formula in (a) from the value of $\zeta(2)$.

Problem 6

Let N be a positive integer. In Problem 6 of Problem Set 2 we saw that the "distribution relation"

$$N^{k-1} \sum_{j=0}^{N-1} f\big((x + j)/N\big) = f(x)$$

was satisfied when $f = B_k$.

(a) Given $n \in \mathbb{Z}$, show that $\sum_{j=0}^{N-1} e^{2\pi n i j/N}$ equals 1 or 0 according as N does or does not divide n. (Hint: Use the formula for $1 + r + r^2 + \cdots + r^{N-1}$ with $r = e^{2\pi i n/N}$.)

(b) Use the Fourier expansion of $\mathbb{B}_k$ to prove that $\mathbb{B}_k$ satisfies the distribution relation for $k \geq 2$ and also for $k = 1$, at least if $x \notin \mathbb{Z}$.

Problem 7

Now give a proof that the distribution relation is satisfied by all $k \geq 1$ and all $x \in \mathbb{R}$ by reducing to the familiar case $f = B_k$ from Problem 6 of Problem Set 2. (Hint: Show that the left-hand of the distribution relation is a periodic function of x with period 1.)

Problem 8

Show that in contrast to B_k, which is not quite an even or odd function (see Problem 2 of Problem Set 2), the function $\mathbb{B}_k$ is even or odd according as k as even or odd, which the provision that if $k = 1$ then we must exclude $x \in \mathbb{Z}$. In other words, $\mathbb{B}_k(-x) = (-1)^k \mathbb{B}_k(x)$ if $k \neq 1$ or $x \notin \mathbb{Z}$.

Problem 9

This problem is a converse to Problems 6 and 7. Let f be a periodic function on $\mathbb{R}$ with a Fourier expansion $f(x) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n x}$, and suppose that f satisfies the distribution relation as well as the parity relation $f(-x) = (-1)^k f(x)$. Show that $f = c\mathbb{B}_k$ for some constant c.

Problem 10

Let $\mathcal{F}$ be the set of $z \in H$ such that $-1/2 \leq x \leq 1/2$ and $|z| \geq 1$. The purpose of this problem is to prove that for every $z \in H$ there exists $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma z \in \mathcal{F}$. Write $x(\gamma z)$ and $y(\gamma z)$ for the real and imaginary parts of γz.

(a) Given $z = x + iy \in H$, show that the set of imaginary parts $y(\gamma z)$ with $\gamma \in SL_2(\mathbb{Z})$ and $y(\gamma z) > y$ is *finite*. (Hint: Since $y(\gamma z) = y/|cz+d|^2$, it is enough to show that there are only finitely many pairs of integers (c, d) such that $|cz+d|^2 < 1$. Now use Problem 12 on Problem Set 2.)

(b) Deduce that the set $\{y(\gamma z) : \gamma \in \Gamma\}$ has a maximal element. Then show that $\gamma \in SL_2(\mathbb{Z})$ can be chosen so that $y(\gamma z)$ is maximal and $-1/2 \leq x(\gamma z) \leq 1/2$. (Hint: Replace γz by $T^n \gamma z$ if necessary, where n is an appropriate integer.)

(c) Show that if γ is chosen as in (b) then $|\gamma z| \geq 1$, whence $\gamma z \in \mathcal{F}$. (Hint: If $|\gamma z| < 1$ show that $y(S\gamma z) > y(\gamma z)$, contradicting the maximality of $y(\gamma z)$.)

4 Week 4

4.1 July 24, 2023

Definition 4.1: Congruence Subgroup

For $N \in \mathbb{N}$, define the **congruence subgroup** $\Gamma_1(N)$ as the subgroup of all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\begin{aligned} a \equiv d \equiv& 1 \mod N \\ c \equiv& 0 \mod N \end{aligned}$$

so $\gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N$.

Definition 4.2: $\Gamma_0(N)$

$\Gamma_0(N)$ is the subgroup of all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $c \equiv 0 \mod N$

so $\gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod N$.

Remark.

If $N = 1$, then $\Gamma_1(N) = SL_2(\mathbb{Z})$.

Before we define some new modular forms, we first start with a terminology that is used.

Definition 4.3: Polynomial Growth

A function $f(n)$ has a polynomial growth if there exists constants c, $d > 0$ such that

- $|f(n)| \leq c \cdot n^d$ for $n > 0$
- $|f(n)| < n^d$ for sufficiently large n.

Definition 4.4: Modular Form for Γ

Let Γ be $\Gamma_1(N)$ or $\Gamma_0(N)$. A modular form of weight k for Γ is a function $f : H \to \mathbb{C}$ such that

1. $f(\gamma z) = (cz + d)^k f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
2. f is represented by a convergent Fourier series
$$f(z) = \sum_{n \geq 0} a(n) e^{2\pi i n z}$$
where $a(n)$ has polynomial growth in n.

Question. In $E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$, does $\sigma_{k-1}(n)$ have polynomial growth?

Solution

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \leq n \cdot n^{k-1} = n^k,$$

so $\sigma_{k-1}(n)$ has polynomial growth.

Definition 4.5: Vector Space of Modular Forms

Define $M_k(N)$ be the **vector space of modular forms** or weight k for $\Gamma_1(N)$.

Example 9

$M_1(N)$ is the vector space of modular forms of weight k for $SL_2(\mathbb{Z})$.

Theorem 4.1: New Modular Forms from Old

If $f \in M_k(N)$ and $g \in M_l(N)$, then $fg \in M_{k+l}(N)$.

Proof. We need to prove that fg satisfies the two properties of modular forms. For property 1, since $f(\gamma z) = (cz+d)^k f(z)$ and $g(\gamma z) = (cz+d)^l g(z)$, $(fg)(\gamma z) = (cz+d)^{k+l}(fg)(z)$. For property 2, you can just multiply the Fourier series of f and g. ■

Corollary

$\mathrm{E}_4^2 = E_8$.

Proof. $E_4 \in M_4(1)$. Then $E_4^2 \in M_8(1)$. Therefore $E_4^2 = cE_8$ for some constant c. If we compare the constant terms of the expansion,

$$\Big(1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n\Big)^2 = c\Big(1 + 480 \sum_{n \geq 1} \sigma_7(n) q^n\Big),$$

so $c = 1$ and $E_4^2 = E_8$. ■

4.2 July 26, 2023

We use a new notation. Let f be a modular form.

Definition 4.6: $(f\,|_k\,\gamma)(z)$

$$(f\,|_k\,\gamma)(z) = \frac{(\det \gamma)^{k/2}}{(cz+d)^k} f(\gamma z).$$

Remark.

If $\gamma = aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then $f\,|_k\,\gamma = f$ because

$$f\,|_k\,\gamma = \frac{a^{2k/2}}{a^k} f(z) = f(z).$$

Remark.

If $\gamma \in SL_2(\mathbb{Z})$, then

$$f\,|_k\,\gamma = (cz+d)^{-k} f(\gamma z) = f(z)$$

by the definition of a modular form.

Let $P = E_2$, $Q = E_4$, and $R = E_6$. Recall that $P = E_2 = 1 - 24\sum_{n\geq 1}\sigma(n)q^n$ where $q = e^{2\pi i z}$ and $\sigma = \sigma_1(n)\sum_{d|n} d$. We have

$$E_2(Sz) = z^2E_2(z) + \frac{12z}{2\pi i},$$

or equivalently, $P\,|_2\,S = P + \rho$ where $\rho = \dfrac{12}{2\pi i z}$.

Definition 4.7: $\upsilon(N)$

Define $\upsilon(N) \in GL_2^+(\mathbb{R})$ by $\upsilon(N) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ where $N > 0$.

Then, $f\,|_k\,\upsilon(N) = \dfrac{N^{k/2}}{(0z+1)^k}f(Nz)$. So if $k = 2$, we get $f\,|_2\,\upsilon(N) = Nf(Nz)$.

We now define D by $D = P\,|\,\upsilon(2) - P$. We will show that $D \in M_2(2)$, i.e. D is a modular form of weight 2 for $\Gamma_1(2)$.

We first look at the Fourier expansion.

Proof. By definition,

$$\begin{aligned}
D &= P\,|\,\upsilon(2) - P \\
&= 2\big(1 - 24\sum_{n\geq 1}\sigma(n)q^n e^{2\pi i n(2z)}\big) - \big(1 - 24\sum_{n\geq 1}\sigma(n)q^n\big) \\
&= 1 + 24\big(\sum_{n\geq 1}\sigma(n)q^n - \sum_{n\geq 1}2\sigma(n)q^{2n}\big) \\
&= 1 + 24\big(\sum_{\substack{n\geq 1\\ n\text{ odd}}}\sigma(n)q^n + \sum_{m\geq 1}\sigma(2m)q^{2m} - \sum_{n\geq 1}\sigma_{\text{even}}(2n)q^{2n}\big) \qquad (n = 2m) \\
&= 1 + 24\big(\sum_{\substack{n\geq 1\\ n\text{ odd}}}\sigma(n)q^n + \sum_{n\geq 1}\sigma_{\text{odd}}(2n)q^{2n}\big) \\
&= 1 + 24\sum_{n\geq 1}\sigma_{\text{odd}}(n)q^n. \qquad \blacksquare
\end{aligned}$$

We now must show that $D\,|_2\,\gamma = D$ for $\gamma \in \Gamma_1(2)$. Recall that $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T = U^N$ generate $\Gamma_1(N)$. So T and U^2 generate $\Gamma_1(2)$. Therefore, we can write

$$\gamma = T^{n_1}U^{2n_2}T_{n_3}\cdots T^{n_l}U^{2n_{l+1}}.$$

If we show that $D\,|\,T = D$ and $D\,|\,U^2 = D$, then we can get $D\,|\,\gamma = D$ for all $\gamma \in \Gamma_1(2)$.

4.3 July 27, 2023

I thank Emmy Huang for helping me with notes for this session.

We begin with defining a new operator.

Definition 4.8: $\omega(N)$

For $N \geq 1$, let $\omega(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Remark.

If $N = 1$, then $\omega(1) = S$.

Lemma

$\omega(N)$ normalizes $\Gamma_1(N)$, i.e. $\omega(N)\Gamma_1(N)\omega(N)^{-1} = \Gamma_1(N)$.

Proof. Suppose $\gamma \in \Gamma_1(N)$. Then

$$\omega(N)\gamma\omega(N)^{-1} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1/N \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -C/N \\ -Nb & a \end{pmatrix}.$$

Since $a \equiv d \equiv 1 \mod N$, and $-Nb \equiv 0 \mod N$, $\omega(N)\Gamma_1(N)\omega(N)^{-1} \subseteq \Gamma_1(N)$. Similar calculation gives $\omega(N)^{-1}\Gamma_1(N)\omega(N) \subseteq \Gamma_1(N)$, so $\Gamma_1(N) \subseteq \omega(N)\Gamma_1(N)\omega(N)^{-1}$. Therefore, $\omega(N)\Gamma_1(N)\omega(N)^{-1} = \Gamma_1(N)$. ■

Corollary

If $f \in M_k(N)$, then $f\,|\,\omega(N) \in M_k(N)$.

Proof. We only prove property 1. Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$,

$$\begin{aligned} (f\,|\,\omega(N))\,|\,\gamma &= f\,|\,\omega(N)\gamma\omega(N)^{-1}\omega(N) \\ &= f\,|\,\omega(N). \end{aligned}$$ ■

Remark.

Recall $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so $T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then

$$\omega(N)T^{-1}\omega(N)^{-1} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1/N \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = U^N.$$

We finally prove property 1 for $D = P\,|\,\upsilon(2) - P$. We must show that $D\,|\,\gamma = D$. Since $\Gamma_1(2)$ is generated by T and U^2, it is enough to show that $D\,|\,T = D$ and $D\,|\,U^2 = D$.

For $D\,|\,T = D$, $D\,|\,T(z) = D(Tz) = D(z+1)$ and since $e^{2\pi i(z+1)} = e^{2\pi iz} \cdot e^{2\pi i} = e^{2\pi iz}$, so any Fourier series is invariant under $z \mapsto z+1$. Therefore, $D\,|\,T(z) = D(z+1) = D(z)$.

Problem Set 4: Fourier expansions and identities

Problem 1

For this week's problem set, it will be useful to have some Bernoulli numbers handy. In class, we saw that $b_0 = 1$, $b_1 = -1/2$, $b_2 = 1/6$, and $b_k = 0$ for all odd $k \geq 3$. Now show that $b_4 = -1/30$, $b_6 = 1/42$, and $b_8 = -1/30$. For the record, $b_10 = 5/66$, but that fact won't be needed in this problem set.

Problem 2

It will also be useful to have a few Fourier expansions of Eisenstein series available. Let $k \geq 4$ be an even integer. Using the formula

$$E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where $\sigma_t(n) = \sum_{d|n} d^t$ and $q = e^{2\pi iz}$, show: $E_4 = 1 + 240(q + 9q^2 + 28q^3 + \cdots)$, $E_6 = 1 - 504(q + 33q^2 + 244q^3 + \cdots)$, and $E_8 = 1 + 480(q + 129q^2 + \cdots)$.

Problem 3

Let $M_k(N)$ be the vector space of modular forms of weight k for $\Gamma_1(N)$. It is a fact that $M_k(1)$ is one-dimensional for $k = 4$, 6, 8, and 10. Using this fact, derive the bizarre identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j).$$

Deduce that if p is a prime then $p^7 = p^3 + 120 \sum_{j=1}^{p-1} \sigma_3(j)\sigma_3(p-j)$.

Problem 4

Let $\mathcal{L} \subset \mathbb{R}^8$ be the set of integral linear combinations of the rows of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix},$$

and put $f(x) = \sum_{l \in \mathcal{L}} e^{l \cdot l \pi i z}$, where $l \cdot l$ is the dot product of l with itself (so if $l = (x_1, x_2, \dots, x_8)$ then $l \cdot l = x_1^2 + x_2^2 + \cdots + x_8^2$).

(a) Prove that $l \cdot l \in 2\mathbb{Z}$ for all $l \in \mathcal{L}$.

(b) Let $a(n)$ be the number of $l \in \mathcal{L}$ such that $l \cdot l = 2n$, so that $f = \sum_{n \geq 0} a(n) q^n$. It is a fact that $f \in M_4(1)$, the space of modular forms of weight 4 for $SL_2(\mathbb{Z})$, which is a one-dimensional space. Deduce that $a(n) = 240\sigma_3(n)$ for $n \geq 1$.

(c) Can you exhibit the 240 points $l \in \mathcal{L}$ such that $l \cdot l = 2$?

Problem 5

Let $\mathcal{L} \in \mathbb{R}^8$ be as in Problem 4, and let $\Lambda \subset \mathbb{R}^{16}$ be the set consisting of vectors $(x_1, x_2, \dots, x_{16})$ such that both $(x_1, x_2, \dots, x_8) \in \mathcal{L}$ and $(x_9, x_{10}, \dots, x_{16}) \in \mathcal{L}$. (One could also write $\Lambda = \mathcal{L} \oplus \mathcal{L}$.) Let $b(n)$ be the number of $\lambda \in \Lambda$ such that $\lambda \cdot \lambda = 2n$. Show that $b(n) = 480\sigma_7(n)$ for ≥ 1.

Problem 6

By comparing E_4^3 with E_6^2, prove that $M_{12}(1)$, the space of modular forms of weight 12 for $SL_2(\mathbb{Z})$, has dimension at least 2. Actually $M_{12}(1)$ has dimension exactly 2, and 12 is the smallest k such that $M_k(1)$ has dimension > 1.

Problem 7

Recall that in Problem 10 of Problem Set 2 we introduced the notation $f\,|_k\,\gamma$ for $\gamma \in GL_2^+(\mathbb{R})$.

(a) Show that $\dfrac{d}{dz}\gamma z = (\det \gamma)(cz+d)^{-2}$.

(b) Let f be a modular form of weight k for $SL_2(\mathbb{Z})$, and take $\gamma \in SL_2(\mathbb{Z})$. By differentiating both sides of the equation $f(\gamma z) = (cz+d)^k f(z)$, show that

$$f'(\gamma z) = (cz+d)^{k+2} f'(z) + kc(cz+d)^{k+1} f(\gamma z).$$

We can write this as $f'\,|_{k+2}\,\gamma = f' + kc(cz+d)^{-1} f(\gamma z)$. Because of the second term on the right-hand side, f' is not quite a modular form of weight $k+2$.

Problem 8

Put $\theta = (2\pi i)^{-1} d/dz$, so that $\theta q^n = \theta e^{2\pi i n z} = nq$. Also, put $P = E_2 = 1 - 24\sum_{n\geq 1} \sigma(n) q^n$, where $\sigma(n) = \sigma_1(n) = \sum_{d|n} d$, and recall that

$$P(Sz) - z^2 P(z) = 12z/(2\pi i).$$

Given a modular form f of weight k for $SL_2(\mathbb{Z})$, put $\partial f = 12\theta f - kPf$. Show that ∂f is a modular form of weight $k+2$ for $SL_2(\mathbb{Z})$. (Hint: It suffices to show that $\partial f\,|_{k+2}\, S = \partial f$, because the existence of a Fourier expansion for ∂f gives $\partial f\,|\,T = \partial f$.)

Problem 9

Show that $\partial E_4 = -4E_6$. (Hint: It suffices to check that the constant terms in the Fourier expansions of the two sides agree. Why?)

Problem 10

Show that ∂ satisfies the product rule for derivatives: If f and g are modular forms for $SL_2(\mathbb{Z})$ of some weights k and l respectively then $\partial(fg) = (\partial f)g + f(\partial g)$.

Problem 11

The purpose of this problem is to prove the identity

$$\sum_{n\in\mathbb{Z}} (w+n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d\geq 1} d^{k-1} e^{2\pi i d w}$$

for $w \in H$ and $k \geq 2$. Recall that the identity above is used to derive the Fourier expansion of $\sum_{(m,n)\neq(0,0)} (mz+n)^{-k}$ for $k \geq 4$ and even.

(a) Show that the identity follows from differentiating both sides of the identity

$$-\pi i + \sum_{n\in\mathbb{Z}} (w+n)^{-1} = (-2\pi i) \sum_{d\geq 1} e^{2\pi i d w}$$

$K-1$ times. (On the left-hand side of the identity in (a) the terms $(w+n)^{-1}$ and $(w-n)^{-1}$ have to be grouped together to ensure absolute convergence.) Thus, after adding $-w^{-1}$ to both sides of the identity in (a) we see that is suffices to prove that

$$-\pi i + \sum_{n\neq 0} (w+n)^{-1} = -2\pi i\left(\frac{1}{2\pi i w} + \sum_{d\geq 1} e^{2\pi i d w}\right).$$

(b) By the "principle of analytic continuation" (a black box, unfortunately) it now suffices to prove that the two sides of the formula above have the same Taylor series expansion at $w = 0$. Express the Taylor series on the left in terms

> of values of $\zeta(s)$ and hence in terms of Bernoulli numbers, and express the Taylor series on the right in terms of the generating function of the Bernoulli numbers.

5 Week 5

5.1 July 31, 2023

To prove $D \,|\, U^2 = D$, we first show that $D \,|\, \omega(2) = -D$. How do we prove $D \,|\, \omega(2) = -D$? To prove this, write

$$\begin{aligned} D \,|\, \omega(2) &= P \,|\, \upsilon(2)\omega(2) - P \,|\, \omega(2) \\ &= P \,|\, S(2I) - P \,|\, S\upsilon(2) \\ &= (P+\rho) - (P+\rho) \,|\, \upsilon(2) && (P \,|\, S = P + \rho, \text{ and ignore } 2I) \\ &= (P+\rho) - \big((P \,|\, \upsilon(2) + \rho \,|\, \upsilon(2)\big) \\ &= (P+\rho) - \big(P \,|\, \upsilon(2) + \rho\big) \\ &= P - P \,|\, \upsilon(2) = -D. \end{aligned}$$

> **Question.** Why is $\rho \,|\, \upsilon(2) = \rho$?

For any f,

$$f\Big|_k \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = r^{k/2} f(z)$$

So let $k = 2$, and we get

$$f\Big|_2 \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = r f(rz).$$

Since $\rho(z) = \dfrac{12}{2\pi i z}$,

$$\rho \,|\, \upsilon(2) = 2 \cdot \frac{12}{2\pi i \cdot 2z} = \frac{12}{2\pi i z} = \rho.$$

> **Claim.** If $\gamma \in \Gamma_1(rN)$, then $\upsilon(r)\gamma\upsilon(r)^{-1} \in \Gamma_1(N)$

Proof. Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(rN)$. Then

$$\begin{aligned} \upsilon(r)\gamma\upsilon(r)^{-1} &= \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 01 & \end{pmatrix} \\ &= \begin{pmatrix} ra & rb \\ c & d \end{pmatrix} \begin{pmatrix} 1/r & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & rb \\ c/r & d \end{pmatrix}. \end{aligned}$$

Since a, $d \equiv 1 \mod rN$, a, $d \equiv \mod N$. Also, since $c \equiv 0 \mod rN$, $c/r \equiv 0 \mod N$. Therefore, $\upsilon(r)\gamma\upsilon(r)^{-1} \in \Gamma_1(N)$. ∎

Claim. If $f \in M_k(N)$ then $f \,|\, \upsilon(r) \in M_k(Nr)$.

Proof. Take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(Nr)$. Then

$$\big(f \,|\, \upsilon(r)\big) \,|\, \gamma = f \,|\, \upsilon(r)\gamma\upsilon(r)^{-1}\upsilon(r) = f \,|\, \upsilon(r)$$

because $\upsilon(r)\gamma\upsilon(r)^{-1} \in \Gamma_1(N)$, hence $f \,|\, \upsilon(r) \in M_k(Nr)$. ∎

Definition 5.1: The Delta Function

Define Δ by

$$\Delta(z) = e^{2\pi i z} \prod_{n\geq 1} (1 - e^{2\pi i n z})^{24} = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

Theorem 5.1

$\Delta \in M_{12}(1)$.

Proof. We start from taking logs from both sides of $\Delta(z) = e^{2\pi i z} \prod_{n\geq 1} (1 - e^{2\pi i n z})^{24}$. ∎

$$\begin{aligned} \log \Delta(z) &= \log e^{2\pi i z} + 24 \sum_{n \geq 1} log(1 - q^n) \\ &= 2\pi i z + 24 \sum_{d \geq 1} \log(1 - q^d). \end{aligned}$$

Take $\dfrac{d}{dz}$ of both sides. Then you get

$$\begin{aligned}
\frac{\Delta'(z)}{\Delta(z)} &= 2\pi i + 24\sum_{d\geq 1}\frac{-e^{2\pi i dz}\cdot 2\pi i d}{1-q^d} \\
&= 2\pi i\left(1 - 24\sum_{d\geq 1} d\frac{q^d}{1-q^d}\right) \\
&= 2\pi i\left(1 - 24\sum_{d\geq 1}\sum_{m\geq 1} d\cdot q^{md}\right) && \left(\frac{r}{1-r} = \sum_{m\geq 1} q^m\right) \\
&= 2\pi i\left(1 - 24\sum_{n\geq 1}\big(\sum_{d|n} d\big)q^n\right) && \text{(Letting } n = md) \\
&= 2\pi i E_2 = 2\pi i P.
\end{aligned}$$

5.2 August 2, 2023

We continue the proof of $\Delta \in M_{12}(1)$. It suffices to show that $\Delta\,|_{12}\, S = \Delta$.

By using the formula $\dfrac{\Delta'(z)}{\Delta(z)} = 2\pi i P$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we get

$$(1z+0)^{-2}\frac{\Delta'(Sz)}{\Delta(Sz)} = \frac{\Delta'}{\Delta}\bigg|_2 S = 2\pi i P\,|_2\, S = 2\pi i(P+\rho)$$

where $\rho = \dfrac{12}{2\pi i z}$.

Since $\dfrac{d}{dz}(Sz) = \dfrac{d}{dz}\left(-\dfrac{1}{z}\right) = z^{-2}$,

$$\text{LHS} = z^{-2}\frac{\Delta'(Sz)}{\Delta(Sz)} = \frac{\frac{d}{dz}\Delta(Sz)}{\Delta(Sz)} = \frac{f'(z)}{f(z)}$$

where $f(z) = \Delta(Sz)$,

and

$$\text{RHS} = 2\pi i P(z) + 2\pi i\rho(z) = \frac{\Delta'(z)}{\Delta(z)} + \frac{12}{z}$$

Since RHS=LHS,

$$\frac{\frac{d}{dz}\Delta(Sz)}{\Delta(Sz)} = \frac{\Delta'(z)}{\Delta(z)} + \frac{12}{z}$$

We take the antiderivative of both sides.

$$\log\Delta(Sz) = \log\Delta(z) + 12\log z + c$$

So
$$\Delta(Sz) = C\Delta(z) \cdot z^{12}$$
where $C = e^c$.

Take $z = i$. Then, $Sz = (0i - 1)/(1i + 0) = i$. So the formula becomes $\Delta(i) = C\Delta(i)$. Since $\Delta(i) \neq 0$ by definition of the delta function, $C = 1$.

> **Question.** In the definition of the delta function, the term in the infinite product is smaller than 1. What happens if we multiply infinite terms less than 1? Does it become 0?

Solution The term goes to 1 as $n \to \infty$, so the infinite product can't go to 0.

So $\Delta(Sz) = \Delta(z)z^{12}$, which is a modular form of weight 12. We will now use Δ to find the dimension of $M_2(4)$. We use $D = 1 + 24\sum_{n\geq 1} \sigma_{\text{odd}}(n)q^n \in M_2(2)$ again.

> **Claim.** If $f \in M_k(N)$ and $r \geq 1$ then $f \in M_k(rN)$.

Proof. Suppose $f \in M_k(N)$ and $\gamma \in \Gamma_1(rN)$. Then, $f\,|_k\,\gamma = f$ because $f \in M_k(N)$. Since $\gamma \in \Gamma_1(rN) \subset \Gamma_1(N)$, so $f \in M_k(rN)$. ■

With $N = 2$ and $r = 2$, we get $D \in M_2(4)$ $(r = 2)$ because $D \in M_2(2)$. Recall that if $f \in M_k(N)$, then $f\,|\,\upsilon(r) \in M_k(Nr)$. Hence $D|\upsilon(2) \in M_2(4)$.
Also recall that $f\,|_k\,\upsilon(r) = r^{k/2}f(rz)$. So for $k = 2$, $f\,|_2\,\upsilon(2) = 2f(2z)$, so $D \in M_2(4)$, and $D^* = \frac{1}{2}D\,|\,\upsilon(2) = D(2z) \in M_2(4)$. Therefore, D, $D^* \in M_2(4)$. Here, $D^* = 1 + 24\sum \sigma_{odd}(n)q^{2n}$.

5.3 August 3, 2023

I thank Diana Harambas for helping me with notes for this session.

We know D, $D^* \in M_2(4)$ where $D = 1+24\sum_{n\geq 1}\sigma_{\text{odd}}(n)q^n$ and $D = 1+24\sum_{n\geq 1}\sigma_{\text{odd}}(n)q^{2n}$.

Thus
$$D = 1 + 24q + 24q^2 + \cdots$$
$$D^* = 1 + 0q + 24q^2 + \cdots$$

So neither D not D^* is a scalar multiple of the other, so these two are linear independent. Since $\dim M_2(4) = 2$, D and D^* are a *basis* for $M_2(4)$. Use the notation $D^* = 1 + 24\sum \sigma_{\text{odd}}(n/2)q^n$, where $\sigma_{\text{odd}}(n/2) = 0$ when n is odd. Now consider
$$\Theta(z) = \sum_{n\geq 1} r_4(n)q^n$$

where $r_4(n) = |\{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \,|\, n_1^2 + n_2^2 + n_3^2 + n_4^2 = n\}|$. Then, $\Theta \in M_2(4)$. Since D and D^* are basis of $M_2(4)$, Θ can be expressed as

$$\Theta = a(1 + 24q + 24q^2 + \cdots) + b(1 + 0q + 24q^2 + \cdots) = 1 + 8q + 24q^2 + \cdots.$$

Thus, since $1 = a + b$ and $8 = 24a$, we get $a = \frac{1}{3}$ and $b = \frac{2}{3}$. So, for $n \geq 1$

$$\begin{aligned} r_4(n) &= \frac{1}{3} \cdot 24\sigma_{\text{odd}}(n) + \frac{2}{3} \cdot 24\sigma_{\text{odd}}(n/2) \\ &= 8\big(\sigma_{\text{odd}}(n) + 2\sigma_{\text{odd}}(n/2)\big) \\ &= 8\Big(\sum_{\substack{d|n \\ d \equiv \pm 1 (4)}} d + \sum_{\substack{d|n \\ d \equiv 2(4)}} d\Big) \\ &= 8\Big(\sum_{\substack{d|n \\ d \not\equiv 0(4)}} d\Big), \end{aligned}$$

so we get Jacobi's formula.

Theorem 5.2: Jacobi's Formula

For $n \geq 1$, $r_4(n) = 8\Big(\sum_{\substack{d|n \\ 4 \nmid d}} d\Big)$.

If k is even, then $M_k(4) = M_k\big(\Gamma_1(4)\big) = M_k\big(\Gamma_0(4)\big)$. Recall that $\Gamma_1(2) = \Gamma_0(2)$.

Lemma

$\mathrm{M}_k\big(\Gamma_1(4)\big) = M_k\big(\Gamma_0(4)\big)$.

Proof. We prove two parts: $M_k\big(\Gamma_1(4)\big) \subseteq M_k\big(\Gamma_0(4)\big)$ and $M_k\big(\Gamma_0(4)\big) \subseteq M_k\big(\Gamma_1(4)\big)$. We certainly have the first part, so we prove the second part. Suppose $f \in M_k\big(\Gamma_1(N)\big)$ and $\gamma \in \Gamma_0(4)$. We want to show that $f\,|_k\,\gamma = f$. We divide into two cases.
First case: If a, $d \equiv 1 \pmod 4$, $\gamma \in \Gamma_1(4)$, so $f \,|\, \gamma = f$.
Second case: If a, $d \not\equiv 1 \pmod 4$, then a, $d \equiv -1 \pmod 4$. Thus

$$\begin{aligned} f \,|\, \gamma &= f \,|\, \gamma(-I)(-I) \\ &= f \,|\, (\gamma(-I)) \,|\, (-I) \\ &= f \,|\, (-I) = f. \end{aligned}$$

■

Problem Set 5: The Delta Function

Problem 1

Recall from Problem 6 on Problem Set 4 that the space of modular forms of weight 12 for $SL_2(\mathbb{Z})$ has dimension 2. Deduce that $E_4^3 - E_6^2 = 1728\Delta$.

Problem 2

This problem involves both some historical background and also a personal note. The historical background is that Hardy once remarked to Ramanujan that 1729 seemed like a boring number, and Ramanujan replied that it wasn't boring at all: It is the smallest positive integer that could be written as a sum of two cubes in two different ways (here "different" means "genuinely different," i.e. not achieved just by switching the order of the summands). The personal note is that once while giving a lecture I carelessly claimed that 1729 is a prime. Somebody in the audience corrected me right away and also pointed out that my integer which can be written as a sum of two cubes in two different ways is not prime.

(a) Factor 1729.

(b) Write 1729 as a sum of two cubes in two different ways.

(c) Prove that if $n > 0$ can be written as a sum of two cubes in two different ways when n is not prime.

Problem 3

When Δ is written as a Fourier series rather than as a product, the Fourier coefficients are usually denoted $\tau(n)$, so that $\Delta = \sum_{n\geq 1} \tau(n)q^n$. Prove the famous Ramanujan congruence $\tau(n) \equiv \sigma_{11}(n) \bmod 691$ for all $n \geq 1$.

Problem 4

The formula $P \,|\, S = P + \rho$ for $P = E_2$ and $\rho(z) = 12/(2\pi i z)$ can be generalized as follows: $(P \,|\, \gamma)(z) = P(z) + 12c/\big(2\pi i(cz+d)\big)$ for arbitrary $\gamma \in SL_2(\mathbb{Z})$. Deduce this formula from the relations $\Delta(\gamma z) = (cz+d)^{12}\Delta(z)$ and $\Delta'/\Delta = 2\pi i P$ by logarithmic differentiation.

Problem 5

This problem leads to a generalization of the modular form D. For an integer $N > 1$ let $\sigma^{(N)}(n)$ be the sum of the positive divisors of n which are relatively prime to N, and put $D_N = -\prod_{p|N}(1-p) + 24\sum_{n\geq 1}\sigma^{(N)}(n)q^n$. So $D_2 = D$. Also $D_N = D_M$, where M is the largest squarefree integer dividing N.

(a) Define $\mu(n)$ to be 0 or $(-1)^t$ according as n is divisible by the square of a prime of $n = p_1p_2\cdots p_t$ with distinct primes $p_1, p_2, \ldots, p_t$. Prove that $\sum_{r|n}\mu(r)$ is 0 or 1 according as $n > 1$ or $n = 1$.

(b) Show that $D_N = -\sum_{r|N} \mu(r) P \,|\, \upsilon(r)$, where $\upsilon(r)$ differs from the identity matrix only in the upper left-hand corner, where 1 is replaced by r.

(c) Deduce that D_N is a modular form of weight 2 for $\Gamma_0(N)$.

Problem 6

Define a twelfth root of Δ by $\Delta^{1/12}(z) = e^{2\pi iz/12} \prod_{n\geq 1}(1 - e^{2\pi inz})^2$.

(a) Show that $\Delta^{1/12} \,|_1\, \gamma = e^{2\pi i\omega(\gamma)/12} \Delta^{1/12}$ for $\gamma \in SL_2(\mathbb{Z})$, where ω is a function from $SL_2(\mathbb{Z})$ to the integers mod 12 which satisfies $\omega(\gamma\delta) = \omega(\gamma) + \omega(\delta)$.

(b) Show that $\log \Delta(-1/z) = 12 \log(z/i) + \log \Delta(z)$.

(c) Prove that $\omega(S) = -3$ and $\omega(T) = 1$, and deduce that ω coincides with the map

$$\omega(\gamma) = (1 - c^2)\big(db + 3d(c - 1) + c + 3\big) + c(d + a - 3) \mod 12.$$

in Problem 10 of Problem Set 1.

(d) Show that $\omega(\gamma) = 0$ for $\gamma \in \Gamma(12)$, where $\Gamma(N)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices which are congruent to the identity matrix modulo N.

Problem 7

Put $f(z) = q \prod_{n\geq 1}(1 - q^n)^2(1 - q^{11n})^2$, where $q = e^{2\pi iz}$ as usual. Show that f is a modular form or weight 2 for $\Gamma_0(11)$. By the way, if we write $f(z) = \sum_{n\geq 1} a(n)q^n$ then for all primes $p \neq 11$ the quantity $p - a(p)$ is the number of solutions (x, y) of the congruence

$$y^2 + y \equiv x^3 - x^2 - 10x - 20 \pmod{p}.$$

For example, if $p = 2$ then there are four solutions, namely $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, and since $a(2) = -2$, we do have $2 - a(2) = 4$.

Problem 8

Let p be an odd prime. Show that any homomorphism $\Gamma_1(p) \to \{\pm 1\}$ is trivial.

Problem 9

Put $f(z) = q \prod_{n\geq 1}(1 - q^n)(1 - q^{23n})$, where $q = e^{2\pi iz}$ as usual. Show that f is a modular form of weight 1 for $\Gamma_1(23)$. (Hint: First prove that f^2 is a modular form of weight 2 for $\Gamma_0(23)$.) By the way, if we write $f(z) = \sum_{n\geq 1} a(n)q^n$ then for all primes $p \neq 23$ we have

$$a(p) = \begin{cases} 0 & \text{if } -23 \text{ is not a square mod } p, \\ 2 & \text{if } p = x^2 + xy + 6y^2 \text{ has a solution with } x,y \in \mathbb{Z} \\ -1 & \text{otherwise.} \end{cases}$$

The first and second cases are mutually exclusive, although this may not be obvious.

Problem 10

It was mentioned in the first week of class that a modular form of weight 0 is constant. This fact can actually be deduced from Problem 10 of Problem Set 3 as follows. Let $\mathcal{F}$ be as in that problem, and $f = \sum_{n \geq 0} a(n) e^{2\pi i n z}$ be a modular form of weight 0 for $SL_2(\mathbb{Z})$. After subtracting of the constant function $a(0)$ from f, we may assume that $a(0) = 0$. The deduction requires two black boxes, unfortunately.

(a) One black box is the fact that a continuous real-valued function on the set $\mathcal{F}^{y_0} = \{z \in \mathcal{F} : y \leq y_0\}$ (or on any "closed and bounded" subset of $\mathbb{C}$) attains a maximum value. So $|f(z)|$ attains a maximum value on $\mathcal{F}^{y_0}$. Using the fact that $\lim_{y \to \infty} |f(z)| = a(0) = 0$, deduce that $|f(z)|$ attains a maximum value on all of $\mathcal{F}$.

(b) Now use Problem 10 of Problem Set 3 to show that f attains a maximum value on all of H, and in fact that the maximum value of $|f(z)|$ on $\mathcal{F}$ is the maximum value $|f(z)|$ on H. Drawing the desired conclusion (that f is constant) requires a second black box: If f is a *holomorphic* function (as our f most certainly is) such that $|f(z)|$ attains a maximum value on a connected open set like H then f is constant. So a modular form of weight 0 for $SL_2(\mathbb{Z})$ is constant.

6 Week 6

6.1 August 7, 2023

Our goal is to prove

$$\dim M_k\big(\Gamma_0(N)\big) \leq 1 + \left\lfloor \frac{kt}{12} \right\rfloor$$

where $t = [SL_2(\mathbb{Z}) : \Gamma_0(N)]$ Note: If $N = 4$, $k = 2$, then $t = 6$. So we get

$$\dim M_2\big(\Gamma_0(4)\big) \leq 2$$

But D, D^* are linear independent and in $M_2\big(\Gamma_0(4)\big)$, So $\dim M_2\big(\Gamma_0(4)\big) = 2$.

Definition 6.1: Right Coset

Let G be a group, and H be a subgroup of G. A **right coset** of H in G is a set of the form

$$Hg = \{hg : h \in H\}$$

where g is any element from G.

Theorem 6.1

For $g,\, g' \in G$, if $Hg \cap Hg' \neq \varnothing$ if and only if $Hg = Hg'$.

Proof.

$$\begin{aligned} Hg \cap Hg' &\Leftrightarrow hg = h'g' \text{ for some } h, h' \in H \\ &\Leftrightarrow g = (h^{-1}h')g' \\ &\Rightarrow Hg = H(h^{-1}h')g' = Hg'. \end{aligned}$$

■

Corollary

G=$\bigcup_j Hg_j$ (disjoint union)

Definition 6.2: Index

If $G = \bigcup_j^t Hg_j$ (disjoint union), then we call

$$t = [G : H]$$

the index of H in G.

Remark.
We say that H has **finite index** in G if $G = \bigcup_j Hg_j$ involves only finitely many j, i.e. if t is not infinite.

Remark.
Suppose we have $G = \bigcup_j Hg_j$. For any $g \in G$, $Gg = \bigcup_j H(g_j g) = G$.

Then, we define this some other coset $Hg_j g$ by $Hg_{\sigma(j)}$ where σ is a permutation of the set $\{1, 2, \ldots, t\}$. So $g_j g = hg_{\sigma(j)}$ for some $h \in H$.

We now go back to $SL_2(\mathbb{Z})$ and $\Gamma_0(N)$. Note that

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

is finite. Now, if $f \in M_k\big(\Gamma_0(N)\big)$, then

$$F = \prod_{j=1}^{t} f \,|_k\, \delta_j \in M_k t\big(SL_2(\mathbb{Z})\big)$$

where $SL_2(\mathbb{Z}) = \bigcup_{j=1}^{t} \Gamma_0(N)\delta_j$ (disjoint union).

Lemma

$F|_{kt}\,\delta = F$.

Proof. Let $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Observe that $(F|_{kt}\,\delta)(z) = (cz+d)^{-kt}F(\delta z)$. So,

$$\begin{aligned} F|_{kt}\,\delta &= (cz+d)^{-kt}F(\delta z) \\ &= (cz+d)^{-kt}\prod_{j=1}^{t}(f|_k\,\delta_j)(\delta z) \\ &= \prod_{j=1}^{t}(cz+d)^{-k}(f|_k\,\delta_j)(\delta z) \\ &= \prod_{j=1}^{t}(f|_k\,\delta_j)|_k\,\delta \\ &= \prod_{j=1}^{t} f|_k\,\delta_j\delta \\ &= \prod_{j=1}^{t} f|_k\,\gamma_j\delta_{\sigma(j)} \end{aligned}$$

where $\gamma_j \in \Gamma_1(N)$. Therefore,

$$\begin{aligned} F|_{kt}\,\delta &= \prod_{j=1}^{t} f\,|\,\gamma_j\delta_{\sigma(j)} \\ &= \prod_{j=1}^{t}(f|_k\,\delta_j)|_k\,\delta_{\sigma(j)} \\ &= \prod_{j=1}^{t} f|_k\,\delta_{\sigma(j)} = F. \end{aligned}$$

■

6.2 August 9, 2023

I thank Vincent Tran for helping me with notes for this session.

Our goal is to prove $\dim M_k\big(\Gamma_0(N)\big) \leq 1 + \lfloor \frac{kt}{12} \rfloor$ where $t = [SL_2(\mathbb{Z} : \Gamma_0(N)]$.

Let $\nu = \lfloor \frac{kt}{12} \rfloor$. Define a linear map

$$T : M_k\big(\Gamma_0(N)\big) \to \mathbb{C}^{\nu+1}.$$

by $T(f) = T\big(\sum_{n\geq 0} a_f(n)q^n\big) = \big(a_f(0), a_f(1), \ldots, a_f(\nu)\big)$.

We now want to show that T is injective. Thus we want to show that if $T(g) = T(h)$ for some g, $h \in M_k\big(\Gamma_0(N)\big)$, then $g = h$. Thus $f = g - h \neq 0$. But $T(f) = 0$, which is a contradiction. Now write $SL_2(\mathbb{Z})$ as a disjoint union $\bigcup_{j=1}^{t} \Gamma_0(N) f_j$. Then $F = {}_{j=1}^{t} f_j \in M_{kt}\big(SL_2(\mathbb{Z})\big)$. So $F = 0 + 0\omega + 0q^2 + 0q^3 + \cdots + q^{\nu+1} + \cdots$ and $F^{12} = 0 + 0q + \cdots + 0q^{(\nu+1)12-1} + q^{(\nu+1)12}$. Since $\Delta = q \prod_{n\geq 1} = 0 + (\text{something})q + (\text{something})q^2$, $\Delta^{kt} = 0 + 0q + 0q^2 + \cdots + (\text{something})q^{kt} + \cdots \in M_{12kt}\big(SL_2(\mathbb{Z})\big)$. As Δ is never zero on the upper half plane, we can consider F^{12}/Δ^{kt}.

Claim. F^{12}/Δ^{kt} is 0.

We then want to find the weight of $\frac{F^{12}}{\Delta^{kt}}(\gamma z) = \frac{(cz+d)^{12kt}F(z)^{12}}{(cz+d)^{12kt}\Delta(z)^{kt}}$. This is weight 0 after cancelling. Since modular forms of weight 0 are constants, F^{12}/Δ^{kt} is a constant. Next, we'll show that the constant is 0. This is a contradiction since if $F = 0$, then $\prod_{j=1}^{t} f\,|_k\, f_j = 0$, so $f = 0$, which is a problem since f is not 0. So T is injective and $\dim M_k\big(\Gamma_0(N)\big) \leq 1 + \nu$. The function f doesn't explode as $y \to \infty$, but if there are negative powers there'd be an explosion. Thus we still must check Fourier expansion.

$$F^{12} = Cq^{(\nu+1)12} + \cdots$$

$$\Delta^{kt} = Cq^{kt}.$$

where C is locally some constant. So

$$\frac{F^{12}}{\Delta^{kt}} = Cq^{(\nu+1)12-kt} + \cdots.$$

In order to ensure that there is no explosion, $(\nu+1)12 - kt$ should be greater or equal than 0 (i.e. the second property of modular forms). If there is equality, then we have that it is a constant. But if we show that $(\nu+1)12 - kt > 0$, then $\lim_{y\to\infty} \frac{F^{12}}{\Delta^{kt}}(iy) = 0$, so the constant is 0. Therefore, our goal is to show that $(\nu+1)12 \geq kt$.

6.3 August 10, 2023

I thank Eamon Zhang for helping me with notes for this session.

Definition 6.3: $\vartheta(t)$

Define $\vartheta(t)$ as

$$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 t} \text{ for } t > 0$$

Observe that $\vartheta(t)^4 = \sum_{n\geq 0} r_4(n)e^{-\pi n t} = \sum_{n\geq 0} r_4(n)e^{(2\pi i n)(it/2)}$. Then $\vartheta(y)^4 = \Theta\left(\frac{iy}{2}\right)$ for $y > 0$.

Claim. $\vartheta(\frac{1}{y}\sqrt{y}\vartheta(y)$.

By principle of analytic cont, $\Theta\,|\,\omega(4) = -\Theta$, $\Theta\,|\,\omega(4)^{-1} = -\Theta$. Now recall that

$$\begin{aligned}\Theta\,|\,U^4 &= \Theta\,|\,\omega(4)\,|\,T^{-4}\,|\,\omega(4)^{-1}\\ &= -\Theta\,|\,T^{-4}\,|\,\omega(4)^{-1}\\ &= -\Theta\,|\,\omega(4)^{-1} = \Theta.\end{aligned}$$

So $\Theta\,|\,U^4 = \Theta$ and also $\Theta\,|\,T = \Theta$. Since T and U^4 generate $\Gamma_1(4)$ and $M_2(4) = M_2\big(\Gamma_1(4)\big) = M_2\big(\Gamma_0(4)\big)$, we get $\Theta \in M_2\Gamma_0(4)$.

Question. Why do we have $\vartheta(1/t) = \sqrt{t}\vartheta(t)$ for $t > 0$?

We use Fourier analysis for this. Recall that $\vartheta(t) = -\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$. For functions $f : \mathbb{R} \to \mathbb{C}$ of *rapid decay*, we define the Fourier transform of f by

$$\hat{f}(x) = \int_0^\infty f(y)e^{-2\pi i x y}\,dy.$$

Then we have some properties.

Theorem 6.2: Poisson Summation Formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f(n)}$$

Lemma

f $f(x) = e^{-\pi x^2}$ then $\hat{f} = f$.

Now define $f_a(x) = f(ax)$ where $a > 0$. Then, $\hat{f}_a(x) = \frac{1}{a}\hat{f}(x/a)$.

Proof.

$$\begin{aligned}
\hat{f}_a(x) &= \int_{-\infty}^{\infty} f_a(y)e^{-2\pi ixy}\,dy \\
&= \int_{-\infty}^{\infty} f(ay)e^{-2\pi ixy}\,dy \\
&= \frac{1}{a}\int_{-\infty}^{\infty} f(ay)e^{-2\pi i}\frac{x}{a}(ay)(ady).
\end{aligned}$$

Let $u = ay$. Then

$$\begin{aligned}
\frac{1}{a}\int_{-\infty}^{\infty} f(ay)e^{-2\pi i}\frac{x}{a}(ay)(ady) &= \frac{1}{a}\int_{-\infty}^{\infty} f(u)e^{-2\pi i}\frac{x}{a}(u)\,du \\
&= \frac{1}{a}\hat{f}\left(\frac{x}{a}\right)
\end{aligned}$$

■

We now apply with $f(x) = e^{-pix^2}$. We get

$$\begin{aligned}
\hat{f}_a(x) = \frac{1}{a}\hat{f}\left(\frac{x}{a}\right) &= \frac{1}{a}e^{\pi(x/a)^2} \\
&= \frac{1}{a}e^{-\pi x^2/a^2}.
\end{aligned}$$

So by the theorem with f_a above, we get

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f_a(n) &= \sum_{n\in\mathbb{Z}} f(an) \\
&= \sum_{n\in\mathbb{Z}} e^{-pin^2a^2} \\
&= \sum_{n\in\mathbb{Z}} \hat{f}_a(n) \\
&= \frac{1}{a}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/a^2}.
\end{aligned}$$

Putting $y = a^2 > 0$, we get the desired result.

Problem Set 6: Farewell!

Problem 1

Recall that $\sigma(n)$ is $\sigma_1(n)$, the sum of the divisors of n. Put $P = 1 - 24\sum_{n\geq 1}\sigma(n)q$ and $\rho(z) = 12/(2\pi iz)$. The purpose of this problem and the two that follow is to prove the identity

$$P \,|\, S = P + \rho \tag{1}$$

where $| = |_2$. Put

$$g(z) = 2\zeta(2) + \sum_{m\neq 0}(\sum_{n\in\mathbb{Z}}(mz+n)^{-2}). \tag{2}$$

(The order of summation is important here, because the sum over (m, n) is not absolutely convergent.) Since $g = (\pi^2/3)P$, the identity $P \,|\, S = P + \rho$ amounts to

$$z^{-2}g(-1/z) = g(z) - (2\pi i)/z. \tag{3}$$

Put $a_{m,n} = (mz+n)^{-2}$ and show that (3) is equivalent to the relation

$$\sum_{n\in\mathbb{Z}}(\sum_{m\neq 0} a_{m,n}) - \sum_{m\neq 0}(\sum_{n\in\mathbb{Z}} a_{m,n}) = -(2\pi i)/z. \tag{4}$$

(Hint: Replace z by $-1/z$ in (2) and multiply both sides by z^{-2}. Then detach the term $n = 0$ from the sum over n and add the term $m = 0$ to the sum over m.

Problem 2

This problem is a continuation of Problem 1. Put

$$b_{m,n} = (mz+n)^{-1}(mz+n+1)^{-1} = (mz+n)^{-1} - (mz+n+1)^{-1}. \tag{5}$$

(a) Show that the double series $\sum_{n\in\mathbb{Z}}(\sum_{m\neq 0} a_{m,n} - b_{m,n})$ converges absolutely. (Hint: Show that $|a_{m,n} - b_{m,n}| \leq |mz+n|^{-3} + |mz+n+1|^{-3}$.)

(b) It follows from (a) that

$$\sum_{n\in\mathbb{Z}}(\sum_{m\neq 0} a_{m,n} - b_{m,n}) = \sum_{m\neq 0}(\sum_{n\in\mathbb{Z}} a_{m,n} - b_{m,n}),$$

whence (4) is equivalent to $\sum_{n\in\mathbb{Z}}(\sum_{m\neq 0} b_{m,n}) - \sum_{m\neq 0}(\sum_{n\in\mathbb{Z}} b_{m,n}) = -(2\pi i)/z$. Show that $\sum_{m\neq 0}(\sum_{n\in\mathbb{Z}} b_{m,n}) = 0$, and deduce that (4) is equivalent to

$$\sum_{n\in\mathbb{Z}}(\sum_{m\neq 0} b_{m,n}) = -(2\pi i)/z. \tag{6}$$

The next problem outlines a proof of (6) and hence completes the proof for

(1).

Problem 3

In this problem we use the notation

$$\cot(z) = i\frac{e^{2iz}+1}{e^{2iz}-1} = i\frac{1+e^{-2iz}}{1-e^{-2iz}} \tag{7}$$

for $z \in \mathbb{C} \setminus (2\pi i)\mathbb{Z}$. Our aim is to prove (6).

(a) Put $c_{m,n} = (m+n/z)^{-1} + (-m+n/z)^{-1}$. Show that (6) is equivalent to

$$\sum_{n\in\mathbb{Z}}(\sum_{m\geq 1} c_{m,n} - c_{m,n+1}) = -2\pi i \tag{8}$$

(Hint: $\sum_{n\in Z}(\sum_{m\neq 0} b_{m,n}) = \sum_{n\in\mathbb{Z}}\left(\sum_{m\geq 1}(b_{m,n}+b_{-m,n})\right)$. Why?)

(b) Prove the identity $\sum_{m\geq 1}(m+w)^{-1} + (-m+w)^{-1} = \pi\cot(\pi w) - w^{-1}$ for $w \in \mathbb{C}\setminus\mathbb{Z}$.

(c) By applying (b) with $w = n/z$ and $w = (n+1)/z$ show that (8) is equivalent to

$$\lim_{n\to\infty}(d_{-n} - d_n) = -2\pi i, \tag{9}$$

where $d_n = \pi\cot(\pi n/z) - z/n$.

(d) Show that $\lim_{n\to\infty} d_{\pm n} = \pm\pi i$ and conclude that (9) does hold. (Hint: If $z \in H$ then $-1/z \in H$ and therefore the real part of $2\pi i n/z$ is negative or positive according as n is negative or positive. Now use the two expressions for $\cot(z)$ in equation (7).)

Problem 4

The point of this problem is to show that $[SL_2(\mathbb{Z}) : \Gamma_0(4)] = 6$, a fact that was needed in our derivation of Jacobi's formula. Fix a prime p.

(a) Show that a vector $(c,d) \in \mathbb{Z}^2$ is the bottom row of a matrix in $SL_2(\mathbb{Z})$ if and only if $\gcd(c,d) = 1$.

(b) Given δ, $\delta' \in SL_2(\mathbb{Z})$ with entries a, b, c, d and a', b', c', d' respectively, show that $\Gamma_0(p^n)\delta = \Gamma_0(p^n)\delta'$ if and only if $cd' - c'd \equiv 0 \bmod p^n$.

(c) With notation as in (b), deduce that $\Gamma_0(p^n)\delta = \Gamma_0(p^n)\delta'$ if and only if there is an integer u prime to p such that $(c', d') = (uc, ud)$.

(d) Deduce that

$$[SL_2(\mathbb{Z}) : \Gamma_0(p^n)] = \frac{p^{2n} - p^{2(n-1)}}{(p-1)p^{n-1}} = p^n(1+1/p).$$

Then take $p = n = 2$ to get the desired value for $[SL_2(\mathbb{Z}) : \Gamma_0(4)]$.

(e) Although we did not need explicit right coset representatives for $\Gamma_0(4)$ in $SL_2(\mathbb{Z})$ for anything we did, show that (c) gives an effective method for choosing a set of coset representatives. For example, show that

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$
$$\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix}$$

is one possible choice.

Problem 5

By using Problem 4 together with the Chinese Remainder Theorem, prove the more general formula

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + 1/p)$$

for any positive integer N.

Problem 6

This problem gives a proof that the only modular form of weight 2 for $SL_2(\mathbb{Z})$ is 0.

(a) Observe that $[SL_2(\mathbb{Z}) : \Gamma_0(2)] = 3$, and deduce that $M_2(2)$ has dimension 1.

(b) Show that the modular form $D = 1 + \sum_{n\geq 1} \sigma_{\text{odd}}(n)q^n$ is *not* invariant under S, and deduce that $M_2(1) = \{0\}$.

7 Supplementary Problems

A function f is called holomorphic at z_0 if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. A meromorphic function is a function that is holomorphic except at a set of isolated points that are the zeros of $1/f$, and they are called the poles of f. The order of a pole of f at z_0 is the smallest integer n such that $(z-z_0)^n f(z)$ is holomorphic at z_0. If f has a zero at z_0, then $\text{ord}_{z_0}(f)$ is defined as $\text{ord}_{z_0}(1/f)$. If f has neither zero nor a pole, then the order equals zero. The order of a modular form f at $i\infty$ is the smallest n such that a_n (nth Fourier coefficient) is nonzero. Define $\mathfrak{F} = \{z \in \mathbb{H} : -\frac{1}{2} \leq x < \frac{1}{2} \text{ and } |z| > 1\} \cup \{z \in \mathbb{H} : |z| = 1 \text{ and } -\frac{1}{2} \leq x \leq 0\}$, where x is the real part of z. Let f be a modular form of weight k, then

$$\text{ord}_{i\infty}(f) + \frac{1}{2}\text{ord}_i(f) + \frac{1}{3}\text{ord}_{e^{2\pi i/3}}(f) + \sum_{\substack{z_0\in\mathfrak{F}\\ z_0\neq i, e^{2\pi i/3}}} \text{ord}_{z_0}(f) = \frac{k}{12}.$$

This is called the Valence formula, which we will use in the next few exercises.

Problem 1

A modular form is a cusp form if the constant Fourier coefficient in its Fourier series expansion is zero. Prove $\Delta(z) = \frac{E_4^3 - E_6^2}{1728}$ is a cusp form of weight 12 that is nowhere vanishing on the upper half-plane.

Problem 2

Use the valence formula to determine the dimension of $M_k(SL_2(\mathbb{Z}))$, denoted M_k, for $k = 0, 2, 4, 6, 8, 10$. Moreover, determine the dimension of S_k (the space of cusp forms of weight k) in terms of the dimension of M_k. Use the Δ modular form to prove $S_k \cong M_{k-12}$.

Problem 3

Use the previous problem to prove $\dim m_k \leq [\frac{k}{12}]$ for $k \equiv 2 \bmod 12$ and is bounded by $[\frac{k}{12}] + 1$ otherwise. Deduce a similar formula for $\dim S_k$. Then determine a basis in terms of $E_4^a E_6^b$ for a, b satisfying certain simple property. Lastly, check that $M = \oplus M_k$ (direct sum) is a ring and is isomorphic to $\mathbb{C}[x, y]$.

Problem 4

Prove the Valence Formula. A simple closed curve is a curve that does not intersect itself and start and end at the same point. If a function is holomorphic on a disk containing a simple closed piecewise smooth curve C, then the integral around C is zero. You may also use the theorem that if f is meromorphic inside and on a simple closed piecewise-smooth curve C and has no zeros or poles on C, then the integral of $f'(z)/f(z)$ around C equals $2\pi i$ times number of zeros minus number of poles, counting multiplicity.

Problem 5

Use the integral representation of the Fourier coefficient to prove that there exists some constant $C > 0$ such that for the nth Fourier coefficient a_n of a cusp form f, $|a_n| \leq C n^{k/2}$, where C only depends on f.

Problem 6

Let $q = e^{2\pi i z}$, and consider the function $f(z) = q \prod_{n \geq 1} (1 - q^n)^{24}$. You may assume the following facts:

1. f has a Fourier series representation.
2. The second Eisenstein series satisfies $E_2(z) = 1 - 24 \sum_{i=1}^{n} \sigma_1(n) q^n$.
3. For all z in the upper half-plane, $E_2(-\frac{1}{z}) = z^2 E_2(z) + \frac{6z}{\pi i}$.

Prove $f = \Delta$. The Fourier coefficients of Δ are called the Ramanujan tau function, and satisfies many amazing identities (e.g. $\tau(n) \equiv \sigma_1 1(n) \bmod 691$).