Topology 1

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1.1 Topological Spaces

Definition 1.1: Topology

Let X be a non-empty set. A **topology** on a set X is a family $\mathcal{T}$ of subsets satisfying

- $\emptyset$ and X are in $\mathcal{T}$.
- The union of any number of sets in $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$.

Note The members of $\mathcal{T}$ are also called *open sets*.

Remark.

For a non-empty set X, the topology on X may not be unique. For these cases, we write $(X, \mathcal{T}_1)$, $(X, \mathcal{T}_2)$, and so on. The space of these topologies are called the **topological space**.

Example 1

Let $X = \{a, b, c\}$. Then

- $\{\{a\}, \{a, c\}, X\}$ is not a topology since $\emptyset$ is not included.
- $\{\emptyset, \{a\}, \{a, c\}, X\}$ is a topology.
- $\{\emptyset, X\}$ is a topology.

Theorem 1.1

Let $\{\mathcal{T}_i\}_{i\in I}$ be a collection of topologies on X. Then, $\bigcap_{i\in I} \mathcal{T}_i$ is also a topology on X.

Proof. We prove that $\bigcap_{i\in I} \mathcal{T}_i$ satisfies the three axioms of a topology.

First, since each $\mathcal{T}_i$ is a topology on X, $\emptyset$ and X is in $\mathcal{T}_i$ for all $i \in I$. So $\emptyset$ and X are also in $\bigcap_{i\in I} \mathcal{T}_i$.

Next, for any set $G_k \in \bigcap_{i\in I} \mathcal{T}_i$ for all $k \in K$, so $G_k \in \mathcal{T}_i$ for all $i \in I$ and $k \in K$.

Therefore, $\bigcup_{k\in K} G_k \in \mathcal{T}_i$ for all $i \in I$, and $\bigcup_{k\in K} G_k \in \bigcap_{i\in I} \mathcal{T}_i$.

Finally, for any set $G_k \in \bigcap_{i\in I} \mathcal{T}_i$ for $k = 1, 2, \dots, n$, so $G_k \in \mathcal{T}_i$ for all $i \in I$ and $k = 1, 2, \dots, n$. Therefore, $\bigcap_{i=1}^{n} G_k \in \mathcal{T}_i$ for all $i \in I$, and $\bigcap_{i=1}^{n} G_k \in \bigcap_{i\in I} \mathcal{T}_i$.

Therefore, $\bigcap_{i\in I} \mathcal{T}_i$ satisfies the three axioms of a topology, $\bigcap_{i\in I} \mathcal{T}_i$ is also a topology.

■

Example 2

Let $X = \{a, b, c\}$, and

- $\mathcal{T}_1 = \big\{\emptyset, \{a\}, \{a, c\}, X\big\}$
- $\mathcal{T}_2 = \big\{\emptyset, \{a\}, \{a, b\}, X\big\}$

are topologies on X. Then, $\mathcal{T}_1 \cap \mathcal{T}_2 = \big\{\emptyset, \{a\}, X\big\}$ is a topology, but $\mathcal{T}_1 \cup \mathcal{T}_2 = \big\{\emptyset, \{a\}, \{c\}, \{a, b\}, X\big\}$ is not a topology.

Remark.

Unlike intersections, the union of topologies need not be a topology.

The three axioms of topology are equivalent to the following two axioms:

Corollary

$\mathcal{T}$ is a topology if and only if

- The union of any number of sets in $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of any finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

Proof. We only have to prove axiom 1 of topology. It suffices to show that

$$\bigcup_{i\in\emptyset} G_i = \emptyset \text{ and } \bigcap_{i\in\emptyset} G_i = X.$$

First, suppose that there exists $p \in X$ such that $p \in \bigcup_{i\in\emptyset} G_i$. Then, there exists some $i \in \emptyset$ such that $p \in G_i$, which is a contradiction. Therefore, $\bigcup_{i\in\emptyset} G_i = \emptyset$.

Next, it is trivial that $\bigcap_{i\in\emptyset} G_i \subseteq X$. Suppose that there is some $p \in X$ such that $p \notin \bigcap_{i\in\emptyset} G_i$. Then, there exists some $i \in \emptyset$ such that $p \notin G_j$, which is a contradiction.

So $X \subseteq \bigcap_{i \in \emptyset} G_i$, and $\bigcap_{i \in \emptyset} G_i = X$. ■

We state four examples of topology:

- Discrete topology,
- Indiscrete topology,
- Cofinite topology,
- Cocountable topology.

Example 3

(X, D) where $D = \{A \subseteq X \mid A \text{ is a subset of } X\}$, or D is the power set of X, is a discrete topology.

Example 4

(X, I) where $I = \{\emptyset, X\}$ is an indiscrete topology.

Example 5

$(X, \mathcal{T})$ where $\mathcal{T} = \{A \in X \mid A = \emptyset \text{ or } A^{\complement} \text{ is finite}\}$ is a cofinite topology. Show that this is actually a topology.

Solution By definition, $\emptyset \in \mathcal{T}$. Also, since $X^{\complement} = \emptyset$ is finite, $X \in \mathcal{T}$.

Now, for any $G_i \in \mathcal{T}$ for $i \in I$, we have $G_i^{\complement}$ is finite. Then, its intersection $\bigcap_{i \in I} G_i^{\complement}$ is also finite. Now,

$$\left(\bigcup_{i \in I} G_i \right)^{\complement} = \bigcap_{i \in I} G_i^{\complement} \subseteq G_i^{\complement}$$

by De Morgan's law, and $\left(\bigcup_{i \in I} G_i \right)^{\complement}$ is finite. Therefore, $\bigcup_{i \in I} G_i \in \mathcal{T}$.

Finally, for any $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, $\left(\bigcap_{i=1}^{n} G_i \right)^{\complement} = \bigcup_{i=1}^{n} G_i^{\complement}$ is finite, so $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$.

Therefore, $(X, \mathcal{T})$ is a topology.

Example 6

$(X, \mathcal{T})$ where $\mathcal{T} = \{A \subseteq X \mid A = \emptyset \text{ or } A^{\complement} \text{ is countable}\}$ is a cocountable topology. Show that this is actually a topology.

We define **countable** as either *finite* or *countably infnite*. For example, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable, but $\mathbb{R}$ or $\mathbb{R} \setminus \mathbb{Q}$ are uncountable.

Solution First, by definition, $\emptyset \in \mathcal{T}$. Also, since $X^{\complement} = \emptyset$ is countable, $X \in \mathcal{T}$.

For any $G_i \in \mathcal{T}$ for $i \in I$, $G_i^{\complement}$ is countable. Then, its intersection $\bigcap_{i \in I} G_i^{\complement}$ is also countable. Now,

$$\left(\bigcup_{i \in I} G_i\right)^{\complement} = \bigcap_{i \in I} G_i^{\complement} \subseteq G_i^{\complement}$$

by De Morgan's law, and $\left(\bigcup_{i \in I} G_i\right)^{\complement}$ is countable. Therefore, $\bigcup_{i \in I} G_i \in \mathcal{T}$.

Finally, for any $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, $\left(\bigcap_{i=1}^{n} G_i\right)^{\complement} = \bigcup_{i=1}^{n} G_i^{\complement}$ is countable, so $\bigcap_{i=1}^{n} G_i \in \mathcal{T}$.

Therefore, $(X, \mathcal{T})$ is a topology.

1.2 Accumulation Points

Throughout this lecture, we let X a topological space, and A a closed set if not explicitly mentioned.

Definition 1.2: Neighborhood

For a point $p \in X$, a set $H \subseteq X$ is called a **neighborhood** of p if there exists an open set G such that $p \in G \subseteq H$. If H is open, then we say that H is an open neighborhood of p.

Definition 1.3: Interior Point

A point $p \in X$ is called an **interior point** of H if there exists an open neighborhood G of p such that $G \subseteq H$.

Definition 1.4: Accumulation Point

Let X be a topological space, and let A a subset of X. A point $p \in X$ is an **accumulation point** of A if and only if for any open neighborhood G of p,

$$A \cap (G \setminus \{p\}) \neq \emptyset.$$

This means that p is an accumulation point of A if for any neighborhood of p, there exists an element of A inside that neighborhood. Here, p need not to be in A.

We write A' as the set of all accumulation points of A.

Example 7

Let $X = \{a, b, c, d\}$, $A = \{a, b\}$, and $\mathcal{T}_1 = \big\{\emptyset, X, \{a, b\}, \{d\}, \{a, b, d\}\big\}$. In $(X, \mathcal{T}_1)$,

Finding A'		
p	G of p	$G \setminus \{p\}$
a	X, $\{a, b\}$, $\{a, b, d\}$	$\{b, c, d\}$, $\{b\}$, $\{b, d\}$
b	X, $\{a, b\}$, $\{a, b, d\}$	$\{a, c, d\}$, $\{a\}$, $\{a, d\}$
c	X	$\{a, b, d\}$
d	X, $\{d\}$, $\{a, b, d\}$	$\{a, b, c\}$, $\emptyset$, $\{a, b\}$

we have $A' = \{a, b, c\}$.

Note If there is a one-point set in a topology, then the point cannot be an accumulation point.

Example 8

Let $X = \{a, b, c, d\}$, $A = \{a, b\}$, and let $\mathcal{T}_2$ be the discrete topology, and $\mathcal{T}_3$ the indiscrete topology.

In $(X, \mathcal{T}_2)$, there are one-point sets for each $p \in X$, namely $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, none of them are accumulation points, and $A' = \emptyset$.

In $(X, \mathcal{T}_3)$, for any point $p \in X$, the open neighborhood of p is only X. Since $(X \setminus \{p\}) \cap A \neq \emptyset$, p is an accumulation point for any p. Therefore, $A' = X$.

1.3 Closed Sets

We simply define closed sets as the reverse of open sets.

Definition 1.5: Closed Set

Let X be a topological space. A subset A of X is a **closed set** if and only if its complement $A^{\complement}$ is an open set.

Example 9

Let $X = \{a, b, c\}$, and

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}.$$

Then, since $\mathcal{T}$ is open, its complement,

$$\{X, \emptyset, \{b, c\}, \{c\}, \{b\}\}$$

is closed.

Theorem 1.2

If $\{A_i\}_{i\in I}$ is a collection of closed subsets of X, then $\bigcap_{i\in I} A_i$ and $\bigcup_{i=1}^{n} A_i$ are closed.

Proof. Since A_i is closed, its complement $A_i^{\complement}$ is open for $i \in I$. Then, by the definition of topology, $\bigcup_{i\in I} A_i^{\complement}$ is open. By De Morgan's law, since

$$\bigcup_{i\in I} A_i^{\complement} = \left(\bigcap_{i\in I} A_i\right)^{\complement}$$

is open, its complement, $\bigcap_{i\in I} A_i$ is closed.

Similarly, by the definition of topology, $\bigcap_{i=1}^{n} A_i^{\complement}$ is open. By De Morgan's law, since

$$\bigcap_{i=1}^{n} A_i^{\complement} = \left(\bigcup_{i=1}^{n} A_i\right)^{\complement}$$

is open, its complement, $\bigcup_{i=1}^{n} A_i$ is closed. ■

Theorem 1.3

A subset (not necessarily closed) A of X is closed if and only if $A' \subseteq A$. That is, any accumulation point of A is in A.

This theorem can be used to determine if a set is closed. Before we prove the theorem, we start with a lemma.

Lemma

A set G is an open subset of X if and only if for any $p \in G$, p is an interior point of G. That is, there is an open neighborhood H of p such that $H \subseteq G$.

Proof. ($\Rightarrow$) Since G is open, for any point $p \in G$, G is an open neighborhood of p. That is, there is an open neighborhood $H = G$ of p.

($\Leftarrow$) For $p \in G$, let H_p the open neighborhood of p satisfying $H_p \subseteq G$. Then,

$$\begin{aligned} G &= \bigcup_{p \in G} p \\ &\subseteq \bigcup_{p \in G} H_p \ (p \in H_p) \\ &\subseteq \bigcup_{p \in G} G \ (H_p \subseteq G) \\ &= G. \end{aligned}$$

This gives $G \subseteq \bigcup_{p \in G} H_p$ and $\bigcup_{p \in G} H_p \subseteq G$, thus $G = \bigcup_{p \in G} H_p$. Therefore, since H_p is open for all p, its union, G, is also open. ∎

We now prove the theorem.

Proof. ($\Rightarrow$)

Claim. For all $p \in A'$, $p \in A$. That is, any accumulation point of A is in A.

We prove by contradiction. Suppose that there exists some $p \in A'$ such that $p \notin A$. Since A is closed, $A^{\complement}$ is an open neighborhood of p. On the other hand, $p \in A'$ implies that for any open neighborhood H of p, $A \cap (H \setminus \{p\}) \neq \emptyset$. Taking $H = A^{\complement}$, we get $A \cap \left(A^{\complement} \setminus \{p\}\right) \neq \emptyset$, which is a contradiction because

$$A \cap \left(A^{\complement} \setminus \{p\}\right) \subseteq A \cap A^{\complement} = \emptyset.$$

Therefore, $p \in A$, and $A' \subseteq A$.

($\Leftarrow$)

> **Note** To show that A is closed, we need to show that $A^{\complement}$ is open. However, $\mathcal{T}$ is not explicitly stated here. This is why we need the lemma.

> **Claim.** For any $p \in A^{\complement}$, there is an open neighborhood of p such that $G \subseteq A^{\complement}$.

For any $p \in A^{\complement}$, by assumption, $p \notin A$, so $p \notin A'$. ($A' \subseteq A$) This implies that there is an open neighborhood G of p such that $A \cap (G \setminus \{p\}) = \emptyset$. Since, $p \notin A$,

$$A \cap (G \setminus \{p\}) = A \cap G = \emptyset.$$

Therefore, $G \subseteq A^{\complement}$. By the lemma, $A^{\complement}$ is open, and hence A is closed. ■

1.4 Closure of a Set

Consider a set B (not necessarily closed). The biggest closed set containing B is X because elements in

$$\mathcal{T} = \{\emptyset, X, \cdots\}$$

are all open, and its complement contains X, which is closed. We now want to consider the smallest closed set containing B.

Definition 1.6: Closure

The **closure** $\bar{A}$ of $A \subseteq X$ is the smallest closed subset containing A.

This can also be defined by

$$\bar{A} = \bigcap_{i \in I} F_i,$$

where $\{F_i\}_{i \in I}$ is a collection of closed subsets containing A. So for any closed subset F containing A,

$$A \subseteq \bar{A} \subseteq F.$$

It is not hard to see that $A = \bar{A}$ if and only if A is closed.

> **Remark.**
> $\bar{A} = A \Leftrightarrow A$ is closed.

> **Example 10**
> Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then the closed sets are $\{X, \emptyset, \{b, c\}, \{a\}\}$. Then,
>
> - $A = \{a\} \Rightarrow \bar{A} = X$
> - $B = \{b\} \Rightarrow \bar{B} = \{b, c\}$
> - $C = \{c\} \Rightarrow \bar{C} = \{c\}$

Example 11

Let $(X, \mathcal{T})$ be the cofinite space. That is, $\mathcal{T} = \{A \subseteq X \mid A = \emptyset \text{ or } A^{\complement} \text{ is finite}\}$.

If X is finite, then since $\mathcal{T}$ contains all subsets of X, X is discrete. Therefore, $\forall A \subseteq X$, so $\bar{A} = A$. Note that in discrete space, every set is open and closed at the same time.

If X is infinite, we divide cases to if A is finite or infinite. Note that in cofinite space, if a subset is finite, then it is closed. This is because

$$\mathcal{T} = \{\emptyset, X, X \setminus \{\text{finite}\}\}$$

consists of open sets, and its complement

$$\{X, \emptyset, \{\text{finite}\}\}$$

consists of closed sets.

If A is infinite, then $\bar{A} = X$.

Lemma

If $A \subseteq B$, then $A' \subseteq B'$.

Proof. We claim that $\forall p \in A'$, $p \in B'$. Since p is an accumulation point of A, there exists an open neighborhood G of p such that $A \cap (G \setminus \{p\}) \neq \emptyset$. Then,

$$A \cap (G \setminus \{p\}) \neq \emptyset \to B \cap (G \setminus \{p\}) \neq \emptyset$$

because $A \subseteq B$. Therefore, p is an accumulation point of B, so $p \in B'$. ■

Theorem 1.4

For any $A \subseteq X$, $\bar{A} = A \cup A'$.

Proof. $(\supseteq)$ By the definition of closure, $A \subseteq \bar{A}$, and $\bar{A}$ is closed. By the lemma, we have

$$A' \subseteq (\bar{A})' = \bar{A},$$

so

$$A' \subseteq \bar{A}.$$

Therefore, $A' \subseteq \bar{A}$ and $A \subseteq \bar{A}$, so $A' \cup A \subseteq \bar{A}$.

$(\subseteq)$

Claim. $A \cup A'$ is closed, i.e. $(A \cup A')^{\complement}$ is open.

We prove that if $\forall p \in (A \cup A')^{\complement}$, there exists an open neighborhood G of p such that $G \subseteq (A \cup A')^{\complement}$. Since $p \in (A \cup A')^{\complement}$, $p \notin A$ and $p \notin A'$. So there is an open neighborhood G of p such that

$$A \cap (G \setminus \{p\}) = \emptyset.$$

Since $A \notin A$, $A \cap (G \setminus \{p\}) = A \cap G = \emptyset$, and $G \subseteq A^{\complement}$.

Now, $\forall q \in G$, since G is an open neighborhood of q,

$$A \cap (G \setminus \{q\}) = \emptyset.$$

This gives $A \cap G = \emptyset$, and since $q \notin A'$, $G \cap A' = \emptyset$. Therefore, $G \subseteq (A')^{\complement}$.

So $G \subseteq (A \cup A')^{\complement}$. Since p is an interior point and p was chosen arbitrarily, we conclude that $(A \cup A')^{\complement}$ is open. ■

Corollary

$p \in \bar{A}$ if and only if for all open neighborhood G of p, $A \cap G \neq \emptyset$.

Proof. ($\Rightarrow$) If $p \in \bar{A}$, then $p \in A$ or $p \in A'$. By definition of A', $A \cap G \neq \emptyset$.

($\Leftarrow$) For some open neighborhood G, if $A \cap G \setminus \{p\} = \emptyset$, then $p \in A$.
If $A \cap G \setminus \{p\} \neq \emptyset$, then p is an interior point of A, so $p \in A'$. Therefore, $p = A \cup A' = \bar{A}$. ■

1.5 Interior, Exterior, Boundary

Definition 1.7: Dense

A set B is called **dense** in X if $\bar{B} = X$.

This means $\forall p \in X$, $p \in B$ or $p \in B'$.

Definition 1.8: Interior

The **interior** of A, $\text{int}(A)$ is the union of all open subsets contained in A.

Let $\{G_i\}_{i \in I}$ be a collection of all open subsets contained in A. Then,

$$\text{int}(A) = \bigcup_{i \in I} G_i.$$

For all open set G contained in A,

$$G \subseteq \text{int}(A) \subseteq A.$$

Therefore, $\text{int}(A)$ is the largest open set contained in A.

Remark.
$\text{int}(A) = A$ if and only if A is open.

Example 12
Let $X = \{a, b, c, d\}$, and $\mathcal{T} = \big\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\big\}$. Then,

- $A = \{a, b, c\} \to \text{int}(A) = \{a, b\}$
- $B = \{b, c\} \to \text{int}(B) = \{b\}$
- $C = \{a\} \to \text{int}(C) = \{a\}$

Theorem 1.5

$\text{int}(A) = \{p \in A \mid p \text{ is an interior point of } A\}$.

Proof. Let $K = \{p \in A \mid p \text{ is an interior point of } A\}$.
$(\subseteq)$

Claim. $\forall p \in \text{int}(A)$, there exists an open neighborhood G of p such that $G \subseteq A$. (p is an interior point of A)

$\forall p \in \text{int}(A)$, $\text{int}(A)$ is an open neighborhood of p contained in A by definition. Therefore, $p \in K$.

$(\supseteq)$ $\forall p \in K$, there exists an open neighborhood G_p such that $G_p \subseteq A$. Then,

$$\begin{aligned} K &= \bigcup_{p \in K} \{p\} \\ &\subseteq \bigcup_{p \in K} G_p \\ &\subseteq \bigcup_{p \in K} A \\ &= A, \end{aligned}$$

so $\displaystyle\bigcup_{p \in K} G_p \supseteq K$ is an open subset contained in A. By the definition of $\text{int}(A)$,

$$K \subseteq \bigcup_{p \in K} G_p \subseteq \text{int}(A).$$

Therefore, $\text{int}(A) = K$. ■

Definition 1.9: Exterior

The **exterior** of a set A, $\text{ext}(A)$ is the interior of $A^{\complement}$. That is, $\text{ext}(A) = \text{int}(A^{\complement})$.

For any A, $\text{int}(A)$ and $\text{ext}(A)$ are disjoint.

Definition 1.10: Boundary

The boundary of A, $\text{b}(A)$ is a set of all points in X not belonging to both $\text{int}(A)$ and $\text{ext}(A)$. That is,

$$\text{b}(A) = X \setminus (\text{int}(A) \cup \text{ext}(A))$$

Example 13

Let $X = \{a, b, c, d, e\}$, and $\mathcal{T} = \big\{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\big\}$. If we let $A = \{b, c, d\}$, then

- $\text{int}(A) = \{c, d\}$
- $\text{ext}(A) = \{a\}$
- $\text{b}(A) = \{b, e\}$
- $\bar{A} = \{b, c, d, e\}$
- $A' = \{b, c, d, e\}$

Remark.
$\overline{A \cap B} \neq \bar{A} \cap \bar{B}$ in general, but $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Theorem 1.6

$\bar{A} = \text{int}(A) \cup \text{b}(A)$.

Proof. Since $\text{int}(A) \cup \text{b}(A) = X - \text{ext}(A)$, it suffices to show that $\bar{A}^{\complement} = \text{int}(A^{\complement})$.

($\subseteq$) $\forall p \in \bar{A}^{\complement}$, ($p \notin \bar{A}$) $\exists$ an open neighborhood G of p such that $A \cap G = \emptyset$ by the (contraposition of the) corollary of theorem 1.4. Therefore, $G \subseteq A^{\complement}$, which implies that p is an interior point of $A^{\complement}$, so $p \in \text{int}(A^{\complement})$.

($\supseteq$) $\forall p \in \text{int}(A^{\complement})$, since p is an interior point of $A^{\complement}$, $\exists$ an open neighborhood G of p such that $G \subseteq A^{\complement}$. Therefore, $A \cap G = \emptyset$, yielding $G \subseteq A^{\complement}$, so $p \in A^{\complement}$. ■

1.6 Neighborhoods and Neighborhood Systems

Definition 1.11: Nowhere Dense

A set B is called **nowhere dense** if $\text{int}(\bar{B}) = \emptyset$.

Definition 1.12: Neighborhood System

The class of all neighborhoods of $p \in X$ is called the **neighborhood system** of p, and denoted by $\mathcal{N}_p$.

Theorem 1.7: Neighborhood Axiom

Let $X \neq \emptyset$. $N : x \in X \mapsto N(x)$: a nonempty collection of subsets of X satisfies

1. $A \in N(x) \to x \in A$
2. $A \subseteq B$ for some $A \in N(x) \to B \in N(x)$
3. $A,\, B \in N(x) \to A \cap B \in N(x)$
4. $A,\, B \in N(x)$ and $A \in N(y)$ for $\forall y \in B \to B \subseteq A$

Theorem 1.8: Kuratowski Closure Axiom

Let $X \neq \emptyset$. Consider $C : \mathcal{P}(X) \to \mathcal{P}(X)$. satisfying

1. $C(\emptyset) = \emptyset$
2. $\forall A \subseteq X,\, A \subseteq C(A)$
3. $\forall A \subseteq X,\, C(C(A)) = C(A)$
4. $\forall A,\, B \subseteq X,\, C(A \cup B) = C(A) \cup C(B)$

Since $C(A) = \bar{A}$, the function that maps A to its closure satisfies the axioms above.

1.7 Sequences

Definition 1.13: Convergence

A **sequence** $\{a_n\}_{n\in\mathbb{N}} \subseteq X$ converges to some point $a \in X$ if for all open neighborhood G of a, $\exists N \in \mathbb{N}$ such that

$$n \geq N \to a_n \in G.$$

Example 14

Let X be the discrete space. Then, $a_n \to a$ as $n \to \infty$ implies that for all open neighborhood G of a, $\exists n \in \mathbb{N}$ such that $a_n \in G$ if $n \geq N$. Set G as the one-point set $\{a\}$. Then, $a_i = a$ if $i > n$. This means since some time, a_n becomes a.

Example 15

Let X be the indiscrete space. Since $G = X$, every sequence in the indiscrete space converges, and its value may not unique.

1.8 Fine and Coarse Topologies

Definition 1.14: Fine and Coarse

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, or $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$.

If $\mathcal{T}_1 \neq \mathcal{T}_2$, we say one is strictly fine (or coarse) than the other.

Definition 1.15: Comparable

We say $\mathcal{T}_1$ and $\mathcal{T}_2$ are **comparable** if one is a subset of other.

Example 16

Let $\mathcal{T}_1$ be the indiscrete topology and $\mathcal{T}_2$ the discrete topology. For any topology $\mathcal{T}$, $\mathcal{T}_1$ is coarser than $\mathcal{T}$, which is coarser than $\mathcal{T}_2$.

1.9 Subspace Topologies

Definition 1.16: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space. For all $Y \subseteq X$, the condition $\mathcal{T}_Y = \{G \cap Y \mid G \in \mathcal{T}\}$ is a topology on Y, which is called the **subspace topology** or the **relative topology**.

Then, $(Y, \mathcal{T}_Y)$ is called a *subspace* of $(X, \mathcal{T})$. We just call $(Y, \mathcal{T}_Y)$ as Y, and $(X, \mathcal{T})$ as X.

Remark.

Y is a subspace of X if and only if $Y \subseteq X$ and for all open set G in Y, there exists an open set H in X such that $G = H \cap Y$.

Then, how do we know that $\mathcal{T}_Y$ is actually a topology? We check the three conditions.

(1): $\emptyset \in \mathcal{T}_Y$ since $\emptyset \in \mathcal{T}$, and $Y \in \mathcal{T}_Y$ since $X \in \mathcal{T}$ and $X \cap Y = Y$.

(2): $\forall G_i \in \mathcal{T}_Y$, $\exists H_i \in \mathcal{T}$ such that $G_i = H_i \cap Y$. Thus,

$$\begin{aligned}\bigcup_{i \in I} G_i &= \bigcup_{i \in I} (H_i \cap Y) \\ &= Y \cap \left(\bigcup_{i \in I} H_i \right) \in \mathcal{T}_Y.\end{aligned}$$

(3): Similarly, we have

$$\begin{aligned}\bigcup_{i=1}^{n} G_i &= \bigcup_{i=1}^{n} (H_i \cap Y) \\ &= Y \cap \left(\bigcup_{i=1}^{n} H_i \right) \in \mathcal{T}_Y.\end{aligned}$$

> **Remark.**
> A set may be open relative to a subspace but not open in the entire space.

2 Bases and Subbases

2.1 Basis

Definition 2.1: Basis

A collection of open sets $\mathcal{B}$ is called a **basis** of $\mathcal{T}$ if

- $\mathcal{B} \subseteq \mathcal{T}$
- $\forall G \in \mathcal{T}$, $\exists \{B_i\} \in \mathcal{B}$ such that $G = \bigcup_i B_i$.

Remark.
The second condition is equivalent to $\forall G \in \mathcal{T}$ and $\forall p \in G$, $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subseteq G$.

Proof. ($\Rightarrow$) For all $G \in \mathcal{T}$, $\exists \{B_i\} \subseteq \mathcal{B}$ such that $G = \bigcup_i B_i$. Therefore, $\forall p \in G$, $\exists B_p \in \{B_i\} \subseteq \mathcal{B}$ such that $p \in B_p \subseteq \bigcup_i B_i = G$.

($\Leftarrow$) We have

$$\begin{aligned} G &= \bigcup_{p \in G} \{p\} \\ &= \bigcup_{p \in G} B_p \\ &= \bigcup_{p \in G} G \\ &= G. \end{aligned}$$

Therefore, $G = \bigcup_{p \in G} B_p$. ■

Example 1
The open intervals form a base for the topology on the line $\mathbb{R}$.

Example 2
The open rectangles in the plane $\mathbb{R}^2$, bounded by sides parallel to the x-axis also form a base $\mathcal{B}$ for the topology on $\mathbb{R}^2$. For, let $G \subseteq \mathbb{R}^2$ be open and $p \in G$. There exists an open disk D_p centered at p with $p \in D_p \subseteq G$. Then

any rectangle $B \in \mathcal{B}$ whose vertices lie on the boundary of D_p satisfies

$$p \in B \subseteq D_p \subseteq G.$$

Remark.

Given a collection $\mathcal{B}$ of subsets of a set X, $\mathcal{B}$ will not always be a basis for some topology on X. Other conditions are also needed.

Theorem 2.1

Let $\mathcal{B}$ be a nonempty collection of subsets of X. Then, $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ if and only if

1. $\exists \{B_i\} \subseteq \mathcal{B}$ such that $X = \bigcup_i B_i$
2. $\forall B_1,\, B_2 \in \mathcal{B},\, B_1 \cap B_2 = \bigcup_i B_i$.

Proof. ($\Rightarrow$) Exercise.

($\Leftarrow$) Let $\mathcal{T} = \{\bigcup_i B_i \mid \{B_i\} \subseteq \mathcal{B}\}$.

We will check two things:

- $\mathcal{T}$ is a topology on X
- $\mathcal{B}$ is a basis of $\mathcal{T}$ (trivial)

We only prove the first one.

We have $X \in \mathcal{T}$ by the first condition, and $\emptyset \in \mathcal{T}$ by setting $I = \emptyset$.

$\forall G_i \in \mathcal{T}$ for $(i \in I)$, $\exists \{B_i\}_{i\in I}$ such that $G_i = \bigcup_{i\in I} B_i$. Therefore, $\bigcup_{i\in I} G_i = \bigcup_i (\bigcup_i B_i) \in \mathcal{T}$ by the definition of $\mathcal{T}$.

$\forall G_1,\, G_2 \in \mathcal{T}$, $\exists \{B_{1_i}\}$ and $\{B_{2_j}\} \subseteq \mathcal{B}$ such that $G_1 = \bigcup_i B_{1_i}$ and $G_2 = \bigcup_j B_{2_j}$. Then, $G_1 \cap G_2 = (\bigcup_i B_{1_i}) \cap \left(\bigcup_j B_{2_j}\right) = \bigcup_{i,j} \left(B_{1_i} \cap B_{2_j}\right)$. By the second condition, $\exists \{B_{ij_k}\} \subseteq \mathcal{B}$ such that $B_{1_i} \cap B_{2_j} = \bigcup_k B_{ij_k}$. Therefore, $G_1 \cap G_2 = \bigcup_{i,j} (\bigcup_k B_{ij_k}) \in \mathcal{T}$. ■

Remark.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases of $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$, respectively. If $\forall B \in \mathcal{B}_1$, $\exists \{B_i^*\} \subseteq \mathcal{B}_2$ such that

$$B = \bigcup_i B_i^*,$$

then $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Proof. $\forall G \in \mathcal{T}_1$, $\exists\{B_i\} \subseteq \mathcal{B}_1$ such that $G = \bigcup_i B_i$. $\forall B_i$, $\exists\{B^*_{i\,j}\} \subseteq \mathcal{B}_2 \subseteq \mathcal{T}_2$ such that $B_i = \bigcup_i B^*_{i\,j}$. Therefore, $G = \bigcup_i \left(\bigcup_i B^*_{i\,j}\right) \in \mathcal{T}_2$. Now, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then $\mathcal{T}_1 \subseteq \mathcal{T}_2$. ■

Example 3

Let

$$\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}.$$

Show that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ in $\mathbb{R}$.

Solution The union of all intervals are $\mathbb{R}$. We now have $(a, b] \cap (c, d]$ either the empty set of another open-closed interval. Therefore, $\exists\{B_i\}$, and $\mathcal{B}$ is a basis.

2.2 Subbasis

Definition 2.2: Subbasis

A collection $\mathcal{S}$ is called a **subbasis** of $\mathcal{T}$ if

1. $\mathcal{S} \subseteq \mathcal{T}$
2. $\left\{\bigcap_{i=1}^{n} S_i \mid \{S_i\} \subseteq \mathcal{S}\right\}$ is a basis of $\mathcal{T}$.

Example 4

The class $\mathcal{S}$ of all infinite open intervals is a subbase for $\mathbb{R}$.

Example 5

The class $\mathcal{S}$ of all infinite open strips is a subbase for $\mathbb{R}^2$.

Theorem 2.2

Let S be a nonempty collection of subsets of X. Then there exists a unique topology having S as a subbasis.

Proof. (Existence) Let $\mathcal{B} = \left\{\bigcap_{i=1}^{n} S_i \mid \{S_i\} \subseteq S\right\}$.

Claim. There exists a topology $\mathcal{T}$ having $\mathcal{B}$ as a basis.

First, $\exists\{B_i\} \subseteq \mathcal{B}$ such that

$$X = \bigcup_i B_i$$

since

$$\bigcap_{i\in\emptyset} S_i = X \in \mathcal{B}.$$

Now, for all B_1, $B_2 \in \mathcal{B}$, $\exists\{B_i\} \subseteq \mathcal{B}$ such that

$$B_1 \cap B_2 = \bigcup_i B_i$$

since $\exists\{S_{1_i}\}$ and $\{S_{2_j}\} \subseteq S$ such that

$$B_1 = \bigcap_{i=1}^{n} S_{1_i} \text{ and } B_2 = \bigcap_{j=1}^{m} S_{2_j},$$

so

$$B_1 \cap B_2 = \left(\bigcap_{i=1}^{n} S_{1_i}\right) \cap \left(\bigcap_{j=1}^{m} S_{2_j}\right) \in \mathcal{B}.$$

Therefore, there exists a topology $\mathcal{T}$ having $\mathcal{B}$ as a basis by the previous theorem.

(Uniqueness) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies generated by $\mathcal{B}$.

Claim. $\mathcal{T}_1 = \mathcal{T}_2$.

Since $\forall B \in \mathcal{B}$, $\exists\{B_i^*\} \subseteq \mathcal{B}$ such that

$$B = \bigcup_i B_i^*$$

because $B = \bigcup_i B_i^*$ where $\{B_i^*\} = \{B\}$. Then, by the remark above, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. But since $\mathcal{T}_2$ is also finer than $\mathcal{T}_1$, $\mathcal{T}_1 = \mathcal{T}_2$. ■

Example 6

Let $\mathcal{S} = \{\{a\}, \{a, c\}\}$. Then,

$$\begin{aligned}\mathcal{B} &= \left\{\bigcap_{i=1}^{n} S_i \mid \{S_i\} \subseteq \mathcal{S}\right\} \\ &= \{X, \{a\}, \{a, c\}\}.\end{aligned}$$

The topology generated by $\mathcal{B}$ is

$$\mathcal{T} = \{\emptyset, X, \{a\}, \{a, c\}\},$$

and this is the unique topology having $\mathcal{S}$ as a subbasis.

Theorem 2.3

Let $\mathcal{S}$ be a class of subsets of a nonempty set X. Then the topology $\mathcal{T}$ on X generated by $\mathcal{S}$ is the intersection of all topologies on X that contain $\mathcal{S}$.

2.3 Local Bases

Definition 2.3: Local Basis

A local basis $\mathcal{B}_p$ at $p \in X$ is a collection of subsets of X satisfying

1. $\forall B \in \mathcal{B}_p$, B is an open neighborhood of p
2. For all open neighborhood G of p, $\exists B_p \in \mathcal{B}_p$ such that $p \in B_p \subseteq G$.

Example 7

Let $X = \{a, b, c, d\}$, and $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{a, b, d\}\}$. Then the local bases of a can be

$$\mathcal{B}_a = \{X, \{a, b\}, \{a, b, d\}\} \text{ or } \{\{a, b\}\}.$$

Remark.

For a point p, the local basis $\mathcal{B}_p$ could not be unique.

Theorem 2.4

Let $\{a_n\} \subseteq X$ be a sequence in X and $p \in X$. Then, $a_n \to p$ as $n \to \infty$ if and only if $\forall B \in \mathcal{B}_i$, $\exists N \in \mathbb{N}$ such that $a_n \in B$ for all $n \geq N$.

Proof. ($\Rightarrow$) Let $\mathcal{B}_p = \{B \in \mathcal{B} \mid p \in B\}$. Then by definition, $\forall B \in \mathcal{B}_p$, B is an open neighborhood of p. Therefore, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $a_n \in B$.

($\Leftarrow$) For any open neighborhood G of p, by the definition of a local basis, $\exists B \in \mathcal{B}_p$ such that $B \subseteq G$. Therefore, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in B \subseteq G$. ■

3 Continuous Functions

Definition 3.1: Continuous

A function is continuous on X if for all open set G in Y, $f^{-1}(G)$ is open in X.

Lemma

Let $\mathcal{B}$ be a basis, and $\mathcal{S}$ be a subbasis. The continuity is equivalent to:

1. $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at every $p \in X$ if for all open neighborhood G of $f(p)$ in Y, $f^{-1}(G)$ is a neighborhood of p in X.
2. $\forall B \in \mathcal{B}$, $f^{-1}(B)$ is open in X
3. $\forall S \in \mathcal{S}$, $f^{-1}(S)$ is open in X
4. For all closed set F in Y, $f^{-1}(F)$ is closed in X.
5. For all subset $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$
6. For all subset $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$ (or $\forall p \in \bar{A}$, $f(p) \in \overline{f(A)}$)
7. For all subset $B \subseteq Y$, $f^{-1}(\text{int}(B)) \subseteq \int(f^{-1}(B))$.

Proof. (1) ($\Rightarrow$) For all open neighborhood G of $f(p)$, $f^{-1}(G)$ is open in X and $p \in f^{-1}(G)$, i.e. $f^{-1}(G)$ is an open neighborhood of p in X.

($\Leftarrow$) For all open set G in Y, $f^{-1}(G)$ is a neighborhood of p in X since G is an open neighborhood of $f(p)$. Then there exists an open set H_p on X such that $p \in H_p \subseteq f^{-1}(G)$. Therefore,

$$\begin{aligned} f^{-1}(G) &= \bigcup_{p \in f^{-1}(G)} \{p\} \\ &\subseteq \bigcup_{p \in f^{-1}(G)} H_p \\ &\subseteq \bigcup_{p \in f^{-1}(G)} f^{-1}(G) \\ &= f^{-1}(G), \end{aligned}$$

yielding $f^{-1}(G) = \bigcup_{p \in f^{-1}(G)} H_p$, so $f^{-1}(G)$ is open in X.

(2) ($\Rightarrow$) Trivial because $\forall B \in \mathcal{B}$, B is open in Y.

($\Leftarrow$) For all open set G in Y, Since $\mathcal{B}$ is a basis of Y, $\exists\{B_i\} \subseteq \mathcal{B}$ such that (fill)

(3) Exercise.

(4) ($\Rightarrow$) For all closed F in Y, since $F^{\complement}$ is open in Y, $f^{-1}(F^{\complement}) = \{f^{-1}(F)\}^{\complement}$ is open in X. Therefore, $f^{-1}(F)$ is closed.

($\Leftarrow$) For all open set G in Y, $G^{\complement}$ is closed in X. Then, $f^{-1}(G^{\complement}) = \{f^{-1}(G)\}^{\complement}$ is closed in X. Therefore, $f^{-1}(G)$ is open in X.

(5) ($\Rightarrow$) For all subset $B \subseteq Y$, by the definition of closure, $B \subseteq \bar{B}$, which gives

$$f^{-1}(B) \subseteq f^{-1}(\bar{B}).$$

By (4), $f^{-1}(\bar{B})$ is closed in X, so $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$.

($\Leftarrow$) For all closed set F in Y, $\overline{f^{-1}(F)} \subseteq f^{-1}(\bar{F}) = f^{-1}(F)$. Therefore, $f^{-1}(F) = \overline{f^{-1}(F)}$, so $f^{-1}(F)$ is closed in X.

(6) ($\Rightarrow$) For all subset $A \subseteq X$, since $f(A) \subseteq \overline{f(A)}$,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

By (4), $f^{-1}(\overline{f(A)})$ is closed in X and thus $\bar{A} \subseteq f^{-1}(\overline{f(A)})$, yielding $f(\bar{A}) \subseteq \overline{f(A)}$.

($\Leftarrow$) For all closed set F in Y, letting $A = f^{-1}(F)$, by (6),

$$f(\bar{A}) \subseteq \overline{f(A)} = \bar{F} = F.$$

Then we have $A \subseteq f^{-1}(f(\bar{A})) \subseteq f^{-1}(F) = A$. Since $A \subseteq \bar{A}$, $A = \bar{A}$, and $A = f^{-1}(F)$ is closed.

(7) ($\Rightarrow$) For all subset $B \subseteq Y$, since $\text{int}(B) \subseteq B$, $f^{-1}(\text{int}(B)) \subseteq f^{-1}(B)$ where $f^{-1}(\text{int}(B))$ is open in X. Therefore, $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$.

($\Leftarrow$) For all open set G in Y, by (7),

$$f^{-1}(G) = f^{-1}(\text{int}(G)) \subseteq \text{int}(f^{-1}(G)).$$

By definition of an interior, $\text{int}(f^{-1}(G)) \subseteq f^{-1}(G)$, i.e. $f^{-1}(G) = \text{int}(f^{-1}(G))$. ■

Definition 3.2: Open and Closed Mapping

- A function $f : X \to Y$ is called **open** if for all open set $H \in X$, $f(H)$ is open in Y.
- A function $f : X \to Y$ is called **closed** if for all closed set F in X, $f(F)$ is closed in Y.

Example 1
Let X be a discrete space. Then every function from X to Y is continuous.

Example 2
Let Y be a discrete space. Then every function from X and Y is an open and close d mapping.

Theorem 3.1

Let $f : X \to Y$. Then,

1. f is a closed mapping if and only if $\forall A \subseteq X$, $\overline{f(A)} \subseteq f(\bar{A})$.
2. f is an open mapping if and only if $\forall B \subseteq X$, $f(\text{int}(B)) \subseteq \text{int}(f(B))$.

Proof. (1) ($\Rightarrow$) For any subset $A \subseteq X$, since $A \subseteq \bar{A}$, $f(A) \subseteq f(\bar{A})$. Since f is a closed mapping, $f(\bar{A})$ is cloed in Y. Therefore, by definition of a closure, $\overline{f(A)} \subseteq f(\bar{A})$.

($\Leftarrow$) For any closed set F in X, by assumption, $\overline{f(F)} \subseteq f(\bar{F}) = f(F)$. Sine $f(F) \subseteq \overline{f(F)}$, $f(F) = \overline{f(F)}$, and $f(F)$ is closed in Y.

(2) Exercise. ■

Definition 3.3: Homeomorphism

A function $f : X \to Y$ is called a **homeomorphism** if

- f is continuous on X
- f is an open mapping
- f is invertible (i.e. f is bijective).

The first two conditions are called *bicontinuous*. If there exists a homeomorphism between X and Y, then we say that X is *homeomorphic* to Y.

Definition 3.4: Topological

Let X satisfy the property P. We say that P is **topological** if every Y homeomorphic to X satisfies the property P.

Note X is called **disconnected** if there exists open sets G and H such that

- G, $H \neq \emptyset$
- $G \cap H = \emptyset$
- $G \cup H = X$.

Example 3

The connectedness is a topological property.

Assume that X is disconnected and Y is an arbitrary topological space homeomorphic to X. Since X is disconnected, there exist open sets G and H in X such that

- $G,\ H \neq \emptyset$
- $G \cap H = \emptyset$
- $G \cup H = X$.

On the other hand, there exists a homeomorphism $f : X \to Y$. Then we have

- $f(G)$ and $f(H)$ are open in Y (since f is open)
- $f(G),\ f(H) \neq \emptyset$ (since $G,\ H \neq \emptyset$)
- $f(G) \cap f(H) = \emptyset$ (since f is injective)
- $f(G) \cup f(H) = Y$ (since f is surjective)

Therefore, Y is disconnected, and connectedness (and disconnectedness) is a topological property.

Definition 3.5: Sequentially Continuous

A function $f : X \to Y$ is called **sequentially continuous** at $p \in X$ if $\forall \{a_n\} \subseteq X$ converging to p, $f(a_n) \to f(p)$ as $n \to \infty$.

Theorem 3.2

If a function $f : X \to Y$ is continuous at $p \in X$, then f is sequentially continuous at $p \in X$.

Proof. By the definition of continuous, for any open set G of $f(p)$ in Y, $f^{-1}(G)$ is a neighborhood of p in X, i.e. there exists an open set H of p such that $p \in H \subseteq f^{-1}(G)$. For H, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in H$. This $f(a_n) \in f(H)$ for all $n \geq N$, which implies that $f(a_n) \to f(p)$ as $n \to \infty$. ■

Remark.

The converse of the theorem is not true in general. That is, f may not be continuous at $p \in X$ even f is sequentially continuous at $p \in X$.

For example, consider $(X, \mathcal{T})$ where $\mathcal{T}$ is a cocountable topology. In $(X, \mathcal{T})$, $a_n \to p$ as $n \to \infty$ implies that $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $a_n = p$. That is, for any open neighborhood G of p, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in G$, i.e. $\{p, a_N, a_{N+1}, \cdots,\} \subseteq G$. Since G^{C} is countable, $G^{\mathsf{C}} \cup (\{a_N, a_{N+1}, \cdots\} \setminus \{p\})$ is countable, so $(G - \{a_N, a_{N+1}, \cdots\}) \cup \{p\}$ is an open neighborhood of p. Let this neighborhood be H. For H, $\exists N_* \in \mathbb{N}$ such that if $n \geq N_*$, then $a_n \in H$. Letting $N' = \max\{N, N_*\}$, we see that $a_n \in G$ and $a_n \in H$ for all $n \geq N'$, which implies that $a_n = p$ for all $n \geq N'$.

4 Topology on the Line and Plane

The real line, $\mathbb{R}$, is an *archimedian ordered field.*

4.1 Open Sets in $\mathbb{R}$

Definition 4.1: Interior Point

Let A be a set of real numbers. A point $p \in A$ is an **interior point** of A if there exists some open interval S_p such that

$$p \in S_p \subseteq A.$$

Definition 4.2: Open Set

The set A is open if each of its points is an interior point.

Example 1

An open interval (a, b), where $a < b$, is an open set.

Example 2

The real line $\mathbb{R}$ and the empty set $\emptyset$ is also an open set.

Example 3

The closed interval $[a, b]$, where $a < b$, is not an open set, because a and b are not interior points.

Theorem 4.1

The union of any number of open sets in $\mathbb{R}$ is open.

Theorem 4.2

The intersection of any finite number of open sets in $\mathbb{R}$ is open.

Example 4

Consider the class of open intervals

$$\left\{ A_n = \left\{ -\frac{1}{n}, \frac{1}{n} \right\} : n \in \mathbb{N} \right\}.$$

Then the infinite intersection $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not an open set.

Remark.
The intersection of any number of open sets in $\mathbb{R}$ need not be open.

4.2 Accumulation Points

Definition 4.3: Accumulation Point

A point $p \in \mathbb{R}$ is an **accumulation point** of A if for all open set G containing p,

$$A \cap (G \setminus \{p\}) \neq \emptyset.$$

Example 5
Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. The point 0 is an accumulation point of A.

Example 6
Every real number $p \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 4.3: Bolzano-Weierstrass

Let A be a bounded, infinite set of real numbers. Then A has at least one accumulation point.

4.3 Closed Sets

Definition 4.4: Closed Sets

A subset A of $\mathbb{R}$ is closed if and only if $A^{\complement}$ is open.

Theorem 4.4

subset A of $\mathbb{R}$ is closed if and only if A contains each of its accumulation points.

Example 7
The closed interval $[a, b]$ is a closed set since its complement, $(-\infty, a) \cup (b, \infty)$ is an open set.

Example 8

The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ is not closed since 0 is an accumulation point of A not belonging to A.

Remark.

Sets may be neither open not closed. For example, consider the half-open interval $(a, b]$.

4.4 Compactness

Definition 4.5: Open Cover

A collection $\{O_\alpha\}$ of open sets is called an **open cover** of a set S if $S \subseteq \bigcup_\alpha O_\alpha$.

Definition 4.6: Compact

A set S is **compact** if every open cover of S is covered by a union of a finite subcover.

Theorem 4.5: Heine-Borel

Every closed and bounded interval $[a, b]$ is compact.

Example 9

Let $\mathcal{G} = \left\{G_n = \left(\frac{1}{n+2}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. Then, $\mathcal{G}$ is an open cover of $A = (0, 1)$, but $\mathcal{G}$ does not have a finite subcover.

4.5 Sequences

Definition 4.7: Sequence

A **sequence** is a function whose domain is $\mathbb{N}$.

We call $s(n)$ or s_n the nth term of the sequence. A sequence is called to be bounded if its range is a bounded set.

Definition 4.8: Convergence

A sequence $\langle a_n \rangle$ converges to $b \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \to |a_n - b| < \epsilon.$$

We write $\lim_{n\to\infty} a_n = b$, $\lim a_n = b$, or $a_n \to b$.

Note The sequence $\langle a_n \rangle$ converges to b if every open set containing b contains all but a finite number of terms of $\langle a_n \rangle$.

4.6 Subsequences

Definition 4.9: Subsequence

If $\langle i_n \rangle$ is a sequence of positive integers such that $i_1 < i_2 < \cdots$, then the sequence

$$\langle a_{i_1}, a_{i_2}, \cdots \rangle$$

is called a **subsequence** of $\langle a_n \rangle$.

Example 10

Let $\langle a_n \rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right\rangle$. The sequence $\left\langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \right\rangle$ is a subsequence of $\langle a_n \rangle$, but $\left\langle 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \cdots \right\rangle$ is not.

Theorem 4.6

Every bounded sequence of real numbers has a convergent subsequence.

4.7 Cauchy Sequences

Definition 4.10: Cauchy Sequences

A sequence $\langle a_n \rangle$ is called a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$m, n \geq n_0 \to |a_m - a_n| < \epsilon.$$

Example 11

If $\langle a_n \rangle$ is a sequence of integers, then it is Cauchy if it is of the form $\langle a_1, a_2, \cdots, a_{n_0}, b, b, b \cdots \rangle$.

Lemma

Every convergent sequence is Cauchy.

Definition 4.11: Completeness

A set A is **complete** if every Cauchy sequence $\langle a_n \in A \rangle$ converges to a point in A.

Example 12

The set of integers is complete by the example above.

Example 13

The set of rational numbers is not complete. For instance, a sequence $\langle 1, 1.4, 1.41, 1.414, \cdots \rangle$ converges to $\sqrt{2} \notin \mathbb{Q}$.

Theorem 4.7: Cauchy

Every Cauchy sequence of real numbers converges to a real number.

Note The theorem above shows that $\mathbb{R}$ is complete.