Sequences

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This handout covers sequences from the very beginning to advanced level. The difficulty jumps from recursion, so don't panic.

1 Sequences

Definition 1.1: Sequence

A **sequence** is a function whose domain is the natural numbers, i.e. $f : \mathbb{N} \to \mathbb{R}$. We use the notation $\{a_n\}$.

2 Arithmetic Sequences

Definition 2.1: Arithmetic Sequence

An **arithmetic sequence** is a sequence whose difference between two consecutive elements are the same, i.e.

$$a_{n+1} - a_n = d$$

for all $n \in \mathbb{N}$.

Then a_n can be expressed as $a_n = a_1 + (n-1)d = a_m + (n-m)d$. This formula can be used to find the difference or the number of terms, i.e.

$$d = \frac{a_n - a_1}{n-1} \text{ and}$$

$$n = \frac{a_n - a_1}{d} + 1.$$

Lemma : Midterm Lemma

A sequence $\{a_n\}$ is arithmetic if and only if for all k and n such that $k < n$,

$$a_{n-k} + a_{n+k} = 2a_n.$$

Theorem : Sum of All Terms of an Arithmetic Sequence

$$\sum_{k=1}^{n} a_k = \frac{1}{2}n(a_1 + a_n).$$

We use the notation S_n for $\sum_{k=1}^{n}$.

Example 1

Express $1 + 2 + \cdots n$ in terms of n.

Example 2 (AMC 10A 2019 P5)

What is the greatest number of consecutive integers whose sum is 45?

Example 3 (AMC 10B 2016 P18)

In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

Example 4 (AMC 10B 2002 P19)

Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \text{ and } a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

Example 5

Let $\{a_n\}$ be an arithmetic sequence. If $a_6 + a_{11} + a_{15} + a_{20} = 32$, find S_{25}.

3 Geometric Sequences

Definition 3.1: Geometric Sequence

A **geometric sequence** is a sequence whose ratio between two consecutive elements are the same, i.e.

$$\frac{a_{n+1}}{a_n} = r$$

for all $n \in \mathbb{N}$.

Then a_n can be expressed as $a_n = a_1 r^{n-1} = a_m r^{n-m}$.

Lemma : Midterm Lemma

A sequence $\{a_n\}$ is geometric if and only if for all k and n such that $k < n$,

$$a_{n-k} \cdot a_{n+k} = (a_n)^2.$$

Theorem : Sum of All Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with common ratio r. Then

$$\sum_{k=1}^{n} a_k = a_1 \cdot \frac{1 - r^{n+1}}{1 - r}.$$

Theorem : Infinite Sum of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with common ratio $|r| < 1$. then

$$\sum_{k=1}^{\infty} a_k = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Example 6 (AIME I 2016 P1)

For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Example 7 (AIME II 2012 P2)

Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

4 Sum Formulas

Make sure to memorize the following sum formulas:

- $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$
- $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n} c = cn$
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Remark.
$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) = \sum_{k=1}^{n} a_k b_k$ is not true in general.

Example 8

Express $\sum_{k=1}^{n} (k^2 + k)$ in terms of n.

5 Telescoping

A *Telesciping series* is a series such that the terms cancel out.

Example 9

Evaluate $\sum_{k=1}^{100} \frac{1}{k(k+1)}$.

Above, we used the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. This method is called the *partial fractions* method. Below states the general cases of partial fractions.

- $\frac{ax+b}{(x+p)(x+q)} = \frac{A}{x+p} + \frac{B}{x+q}$
- $\frac{ax^2+bx+c}{(x+p)(x+q)^2} = \frac{A}{x+p} + \frac{B}{x+q} + \frac{C}{(x+q)^2}$
- $\frac{ax^2+bx+c}{(x+p)(x^2+qx+r)} = \frac{A}{x+p} + \frac{Bx+C}{x^2+qx+r}$

Example 10 (AMC 12B 2019 P8)

Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)?$$

Example 11 (AIME I 2002 P4)

Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29}$, for positive integers m and n with $m < n$, find $m+n$.

6 Recursions

Recursion problems in competitions are divided into two cases:

- Linear recurrences
- Nonlinear recurrences.

For linear recurrences, there is a way of finding the general term a_n. This is not true for nonlinear recurrences. This fact makes nonlinear recurrences look somewhat harder, but don't worry! In intermediate-level competitions, the problem won't ask you for the general formula.

Linear Recurrence

Let $\{a_n\}$ be a sequence. A linear recurrence has the form

$$a_n + pa_{n-1} + qa_{n-2} = 0,$$

where p and q are real numbers (but they will usually be integers).

> **Example 12**
> Find the number of ways to fill a 2×10 rectangle with dominoes.

Even though we didn't look for the general formula here, this gave the linear recurrence $a_n - a_{n-1} - a_{n-2} = 0$.

We will now look how to find the general formula. For a linear recurrence

$$a_n + pa_{n-1} + qa_{n-2} = 0,$$

the *characteristic polynomial* is defined as

$$P(x) = x^2 + px + q.$$

Let α and β be distinct roots of the equation $P(x) = 0$. (Note that α and β can be complex numbers.) Then the general term can be expressed by

$$a_n = a\alpha^n + b\beta^n$$

where a and b are constants to find out.

> **Example 13**
> The Fibonacci sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$. Find the general term of $\{a_n\}$.

Note that this method can be done also for higher-orders. For example, if

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3},$$

then the characteristic polynomial becomes

$$P(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3),$$

so $a_n = a \cdot 1^n + b \cdot 2^n + c \cdot 3^n$.

If the characteristic equation has multiple roots, then the setup is a little different. Suppose the characteristic equation is

$$P(x) = (x-\alpha)^2(x-\beta).$$

Then the general term can be expressed by

$$a_n = (ax+b)\alpha^n + c\beta^n.$$

This can also be done for higher orders. If a root γ has multiplicity n, the general formula will include $(a_{n-1}x^{n-1} + \cdots + a_1x + a_0)\gamma^n$.

For problems such that the characteristic equation is not factorizable, don't worry! You won't need to find the general formula for a_n.

Example 14 (AMC 12A 2007 P25)

Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

Example 15 (AIME II 2015 P12)

There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

Nonlinear Recurrence

Every recursion that is not linear is considered nonlinear. This includes some fractional recurrences and multiple recursions. This further divides to two types:

- If the problem asks you for some a_k where k is not large
- If the problem asks you for some a_k where k is large

The first one is a lot easier since some calculations will give you a_k. For the second part, mostly you should look for the general formula of a_n. Two techniques can be used here:

- Change the recursion to a linear recurrence
- Look for repeating terms or telescoping terms

We first focus on the first case where k is not large. The main part (like 99%) of these problems are finding the recursive formula of a_n.

Example 16 (AMC 12A 2019 P23)

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Example 17 (AIME I 2008 P11)

Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

If the problem asks you for some a_k where k is large, you may need to find the general formula of a_n. Since this is way harder than linear recurrence, we change the recursion formula into a linear recurrence. Appropriate substitution will change the recursive formula into a linear recurrence, which is what we want.

Example 18 (AMC 12A 2019 P9)

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive inegers. What is $p + q$?

Example 19

Let $\{a_n\}$ be a sequence such that $a_1 = 1$ and

$$a_{n+1} = \frac{a_n + \sqrt{3}}{1 - \sqrt{3}a_n}.$$

Find a_{1000}.

Example 20 (AIME II 2008 P6)

The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\frac{b_{32}}{a_{32}}$.

We also can look for repeating terms. For example, if we find out that $a_3 = a_6$ with a recursion formula, we can deduce that a_n is periodic with period 3 (or 1 but probably not).

Example 21 (AIME II 2020 P6)

Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Example 22 (AIME II 2011 P12)

Let $f_1(x) = \dfrac{2}{3} - \dfrac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.