Systems of Equations

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If we are given a linear system of equations, it wouldn't take us long to solve, just bash. However, some systems of equations are not like this, and we need to think to find the answer. Here, we go over some techniques we can use.

1 Symmetric and Cyclic Equations

Systems of equations that are symmetric (or cyclic) can be divided into two cases. But before we start, what even does *symmetric* mean? There is no clear definition. If the system somehow looks symmetric (or cyclic), we can try some of the techniques here.

One example of a symmetric system is if the system is of the form

$$f(x,y) = \text{something}$$

$$f(y,x) = \text{something.}$$

Some Identities

Make sure to keep these notable (but nontrivial) identities in mind.

- $x^2+y^2+z^2-xy-yz-zx = \frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)$
- $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$
- $(x+y+z)(x^2+y^2+z^2) = (x^3+y^3+z^3)+(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)$
- $(x+y+z)(xy+yz+zx) = 3xyz+(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)$
- $(x+y)(y+z)(z+x) = 2xyz+(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)$

We now go on to two cases of solving the problem.

Integers

If the solutions to the system are integers, then we may try factoring.

1. Try adding or subtracting the equations.
2. Try to factor the equation.
3. Look for the relation between the variables.
4. Find the solution.

Example 1 (PCMM 2018 High Schol P22)

Positive integers a and b satisfy $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

Walkthrough David Altizio loves systems of equations

1. How should we get rid of c?
2. Eliminate c, and factor.
3. Try cases and finish the problem.

Now, here is a cyclic example.

Example 2 (PCMM 2013 High School P30)

Suppose x, y, and z are integers that satisfy the system of equations

$$x^2y + y^2z + z^2x = 2186$$
$$xy^2 + yz^2 + zx^2 = 2188.$$

Evaluate $x^2 + y^2 + z^2$.

Walkthrough Just say that this is cyclic.

1. 2186 and 2188 look somehow close.
2. Factorize, and use the fact that the constant has only two positive divisors.
3. Finish the problem.

Reals / Complex

If the solutions to the system are reals, then the problem gets harder. This is because for example, $ab = -1$ doesn't guarantee $a = 1$ and $b = -1$ like what we did in the integers.

For this case, one still can try factoring, but now the constant term should be zero. This means, try to make the form $(\text{stuff}) \cdots (\text{blah}) = 0$.

Example 3 (PCMM 2014 High School P19)

Let x, y, and z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$.

Walkthrough Why those random numbers there

1. Try to find the relations between the coefficients.
2. Add the three equations.
3. Finish the problem.

Example 4 (TheCALT 2020 Round1 Target P4)

Suppose u and v are complex numbers satisfying the system of equations

$$(u-1)(v-1) = 9 \qquad \text{and} \qquad u^3 - u^2 = v^3 - v^2.$$

Find the sum of all possible values of $|u|^2 + |v|^2$.

Walkthrough The right equation doesn't have any constants.

1. Try to factor the second equation.
2. Substitute $u + v$ and uv.
3. Solve the system of equations of $u + v$ and uv.
4. Finish the problem.

2 Substitutions

Not only in systems, substitutions are a nice way to solve equations. Unfortunately, there isn't a specific way of taking substitutions. Just read the problem, and if you think taking a substitution is necessary, then find it. Still, some useful substitutions are:

- $x = kx'$, $y = ly'$, $z = mz'$ to make the equation's coefficient consistent
- $x = \frac{1}{a}$, $y = \frac{1}{b}$, and $z = \frac{1}{c}$ (useful when $x+y+z = xyz$ or $xy+yz+zx = xyz$)
- $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$ when $xyz = 1$
- Trigonometric substitutions: $\sin\theta$ or $\cos\theta$ when the range is limited and $\tan\theta$ when the range is not limited
- $x = a+b$, $y = b+c$, $z = c+a$ if x, y, z is somehow related to the three sides of a triangle
- $x+y+z = a$, $xy+yz+zx = b$, $xyz = c$ or some Vieta-related substitutions, however this will not be covered in this handout

Other very clever substitutions can be made!

Example 5 (2022 AIME I P15)
Let x, y, and z be positive real numbers satisfying the system of equations:

$$\sqrt{2x - xy} + \sqrt{2y - xy} = 1$$
$$\sqrt{2y - yz} + \sqrt{2z - yz} = \sqrt{2}$$
$$\sqrt{2z - zx} + \sqrt{2x - zx} = \sqrt{3}.$$

Then $[(1-x)(1-y)(1-z)]^2$ can be written as m/n, where m and n are relatively prime positive integers. Find $m+n$.

Walkthrough It seems like it is hard to find the explicit solution. We look for an alternative approach.

1. Split the equation to

$$\sqrt{x} \cdot \sqrt{2-y} + \sqrt{y} \cdot \sqrt{2-x} = 1$$
$$\sqrt{y} \cdot \sqrt{2-z} + \sqrt{z} \cdot \sqrt{2-y} = \sqrt{2}$$
$$\sqrt{z} \cdot \sqrt{2-x} + \sqrt{x} \cdot \sqrt{2-z} = \sqrt{3}.$$

2. Look for the range that x, y, and z can take.

3. Make an appropriate substitution. (Hint: $\sin^2 + \cos^2 = 1$)

4. Simplify the equation by taking off the square roots. What does the equation seem like?

5. Solve the equation.

6. Finish the problem.

3 Other Techniques

Geometry

Some substitutions can change the problem from algebra to geometry.

Example 6 (2006 AIME II P15)

Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}},$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

Walkthrough Surely a geometry problem.

1. Let x, y, z be the three sides of the triangle.

2. Find the ratio between x, y, and z.

3. Make an equation with the area.

4. Finish the problem.

Using Complex Numbers

Using complex numbers can be helpful. Try multiplying a complex number by an equation. If the problem seems like factorable but it's not, try factorizing over the complex numbers. The following example will help in a better understanding.

Example 7 (CMIMC 2016 Algebra Tiebreaker P3)

Suppose x and y are real numbers which satisfy the system of equations

$$x^2 - 3y^2 = \frac{17}{x} \qquad \text{and} \qquad 3x^2 - y^2 = \frac{23}{y}.$$

Find $x^2 + y^2$.

Walkthrough First, try to solve the problem without the walkthroughs.

1. The constants seems inconsistent. Get rid of it.
2. Multiply i to one of the equations.
3. Change the formula to the power of a complex number.
4. What does $x^2 + y^2$ represent?
5. Finish the problem. The answer should be a cube root of a positive integer.

4 Solutions to Examples

Example 1 (PCMM 2018 High Schol P22)

Positive integers a and b satisfy $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

Solution Subtracting the second equation from the first gives

$$\begin{aligned}(a^3 + 32b + 2c) - (b^3 + 32a + 2c) &= a^3 - b^3 - 32a + 32b \\ &= (a-b)(a^2 + ab + b^2 - 32) = 903.\end{aligned}$$

Since a and b are integers, the cases are

$$(a - b, a^2 + ab + b^2 - 32) = (1, 903), (3, 301), (7, 129), (21, 43), \cdots.$$

Trying each case, we get that $(3, 301)$ works, and $(a, b) = (12, 9)$. Plugging in gives $c = 1$, so $a^2 + b^2 + c^2 = 226$.

Example 2 (PCMM 2013 High School P30)

Suppose x, y, and z are integers that satisfy the system of equations

$$x^2y + y^2z + z^2x = 2186$$
$$xy^2 + yz^2 + zx^2 = 2188.$$

Evaluate $x^2 + y^2 + z^2$.

Solution Subtracting the second equation from first, we get

$$\begin{aligned}(x^2y + y^2z + z^2x) - (xy^2 + yz^2 + zx^2) &= x^2y + y^2z + z^2x - xy^2 - yz^2 - zx^2 \\ &= (x-y)(xy - zx - xy + z^2) \\ &= -(x-y)(y-z)(z-x) = -2.\end{aligned}$$

Therefore, $(x-y)(y-z)(z-x) = 2$. This gives $x-y = -1$, $y-z = -1$, and $z-x = 2$ because $(x-y)+(y-z)+(z-x) = 0$. Plugging in, we get $(x, y, z) = (8, 9, 10)$, and $x^2 + y^2 + z^2 = 245$.

Example 3 (PCMM 2014 High School P19)

Let x, y, and z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$.

Solution Adding three equations, we get

$$\begin{aligned}&x(y^2 + yz + z^2) + y(z^2 + zx + x^2) + z(x^2 + xy + y^2) \\ &= x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 + 3xyz \\ &= (x+y+z)(xy+yz+zx) \\ &= (3y + 10z) + (21z + 24x) + (7x + 28y) = 31(x+y+z).\end{aligned}$$

Since $x + y + z > 0$, $xy + yz + zx = 31$.

Example 4 (TheCALT 2020 Round1 Target P4)

Suppose u and v are complex numbers satisfying the system of equations

$$(u-1)(v-1)=9 \qquad \text{and} \qquad u^3-u^2=v^3-v^2.$$

Find the sum of all possible values of $|u|^2+|v|^2$.

Solution Rearranging the second equation, we get

$$u^3-v^3=u^2-v^2$$
$$(u-v)(u^2+uv+v^2)=(u-v)(u+v).$$

Therefore, $u=v$ or $u^2+uv+v^2=u+v$.

If $u=v$, then $(u-1)^2=9$, so $(u,v)=(-2,-2)$ and $(4,4)$.

If $u^2+uv+v^2=u+v$, since $uv=u+v+8$ from the first equation, we get $u^2+v^2=-8$. Let $u+v=a$, and $uv=b$. Then we get a system of equations

$$-a+b=8$$
$$a^2-b=-8.$$

Solving for this(substitute $a=-b+8$), we get $(a,b)=(-2,6)$ and $(4,12)$. Then, u and v become the two roots to the equations

$$x^2-4x+12=0 \text{ and}$$
$$x^2+2x+6=0.$$

Therefore, $(u,v)=(2+\sqrt{8}i,2-\sqrt{8}i)$ and $(-1+\sqrt{5}i,-1-\sqrt{5}i)$. The desired value is 76.

Example 5 (2022 AIME I P15)

Let x, y, and z be positive real numbers satisfying the system of equations:

$$\sqrt{2x-xy}+\sqrt{2y-xy}=1$$
$$\sqrt{2y-yz}+\sqrt{2z-yz}=\sqrt{2}$$
$$\sqrt{2z-zx}+\sqrt{2x-zx}=\sqrt{3}.$$

Then $[(1-x)(1-y)(1-z)]^2$ can be written as m/n, where m and n are relatively prime positive integers. Find $m+n$.

Solution Split the equation to

$$\sqrt{x}\cdot\sqrt{2-y}+\sqrt{y}\cdot\sqrt{2-x}=1$$
$$\sqrt{y}\cdot\sqrt{2-z}+\sqrt{z}\cdot\sqrt{2-y}=\sqrt{2}$$
$$\sqrt{z}\cdot\sqrt{2-x}+\sqrt{x}\cdot\sqrt{2-z}=\sqrt{3}.$$

Since x, y, and z are between 0 and 2, we make a substitution. Let $x=2\cos^2\alpha$, $y=2\cos^2\beta$, and $z=2\cos^2\theta$. Taking out $\sqrt{2}$s and simplifying, we get

$$\cos\alpha\cdot\sin\beta+\cos\beta\cdot\sin\alpha=\frac{1}{2}$$
$$\cos\beta\cdot\sin\theta+\cos\theta\cdot\sin\beta=\frac{\sqrt{2}}{2}$$
$$\cos\theta\cdot\sin\alpha+\cos\alpha\cdot\sin\theta=\frac{\sqrt{3}}{2}.$$

With the sum formula, we end up with

$$\sin(\alpha+\beta)=\frac{1}{2}$$
$$\sin(\beta+\theta)=\frac{\sqrt{2}}{2}$$
$$\sin(\theta+\alpha)=\frac{\sqrt{3}}{2}.$$

Taking inverse sine, we get

$$\alpha+\beta=\frac{\pi}{6}$$
$$\beta+\theta=\frac{\pi}{4}$$
$$\theta+\alpha=\frac{\pi}{3},$$

and therefore

$$\alpha=\frac{\pi}{8},\ \beta=\frac{\pi}{24},\text{ and }\theta=\frac{5\pi}{24}.$$

Substituting back gives

$$x=2\cos^2\frac{\pi}{8},\ y=2\cos^2\frac{\pi}{24},\text{ and }z=2\cos^2\frac{5\pi}{24}.$$

Now, the desired value is

$$\begin{aligned}[(1-x)(1-y)(1-z)]^2 &= \left[\left(1-2\cos^2\frac{\pi}{8}\right)\left(1-2\cos^2\frac{\pi}{24}\right)\left(1-2\cos^2\frac{5\pi}{24}\right)\right]^2 \\ &= \left[(-1)^3\left(cos\frac{\pi}{4}\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\right)\right]^2 \\ &= \left[(-1)\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}\cdot\frac{\sqrt{6}-\sqrt{2}}{4}\right]^2 = \frac{1}{32}\end{aligned}$$

Therefore, $m+n = 1+32 = 33$.

Example 6 (2006 AIME II P15)

Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}},$$

and that $x+y+z = \dfrac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m+n$.

Solution Consider a triangle ABC with $\overline{BC} = x$, $\overline{CA} = y$, and $\overline{AB} = z$. Define h_x to be the altitude from A to $\overline{BC}$, and h_y, h_z similarly. Then, we have

$$x = \sqrt{y^2 - h_x^2} + \sqrt{z^2 - h_x^2}$$

$$y = \sqrt{z^2 - h_y^2} + \sqrt{x^2 - h_y^2}$$

$$z = \sqrt{x^2 - h_z^2} + \sqrt{y^2 - h_z^2}.$$

So we have $h_x = 1/4$, $h_y = 1/5$, and $h_z = 1/6$. Since

$$S_{\triangle ABC} = \frac{1}{2}xh_x = \frac{1}{2}yh_y = \frac{1}{2}zh_z,$$

$x : y : z = 4 : 5 : 6$. Let $x = 4k$, $y = 5k$ and $z = 6k$. Then, $S_{\triangle ABC} = \dfrac{1}{2}k$.

Now, by Heron's formula,

$$\begin{aligned} S_{\triangle ABC} &= \sqrt{\frac{15}{2}k\left(\frac{15}{2}k-4k\right)\left(\frac{15}{2}k-5k\right)\left(\frac{15}{2}k-6k\right)} \\ &= \frac{k^2}{4}\sqrt{15(15-8)(15-10)(15-12)} \\ &= \frac{15\sqrt{7}}{4}k^2 = \frac{1}{2}k. \end{aligned}$$

Therefore, $\frac{15\sqrt{7}}{4}k = \frac{1}{2}$, and $x+y+z = 15k = \frac{2}{\sqrt{7}}$. This gives $m+n = 2+7 = 9$.

Example 7 (CMIMC 2016 Algebra Tiebreaker P3)

Suppose x and y are real numbers which satisfy the system of equations

$$x^2 - 3y^2 = \frac{17}{x} \qquad \text{and} \qquad 3x^2 - y^2 = \frac{23}{y}.$$

Find $x^2 + y^2$.

Solution Multiplying x in the first equation and y in the second equation, we get

$$x^3 - 3xy^2 = 17 \qquad \text{and} \qquad 3x^2y - y^3 = 23.$$

Multiply i in the second equation and add to the first, we get

$$\begin{aligned} (x^3 - 3xy^2) + i(3x^2y - y^3) &= x^3 + 3x^2yi - 3xy^2 - y^3i \\ &= (x+yi)^3 = 17 + 23i. \end{aligned}$$

Taking the magnitude of each side,

$$\begin{aligned} \sqrt{x^2+y^2}^3 &= \sqrt{17^2+23^2} \\ x^2+y^2 &= \sqrt[3]{17^2+23^2}. \end{aligned}$$

Therefore, $x^2 + y^2 = \sqrt[3]{818}$.

5 Problems

Problem 1 (AMC 12A 2021 P21)

Let a, b, and c be positive integers with $a \geq b \geq c$ such that

$$a^2 - b^2 - c^2 + ab = 2011 \text{ and}$$
$$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997.$$

What is a?

Problem 2 (PCMM 2020 High School P28)

Let p, q, and r be prime numbers such that $2pqr + p + q + r = 2020$. Find $pq + qr + rp$.

Problem 3 (ARML 2007 Team Round P7)

Let S be the set of all complex ordered triple solutions to the following system:

$$x + yz = 7$$
$$y + zx = 10$$
$$z + xy = 10.$$

Let $\left(\hat{X}, \hat{Y}, \hat{Z}\right)$ be the coordinatewise sum of all ordered triples (x, y, z) in S. Compute $\hat{X}$.

Problem 4 (NICE Spring 2021 P13)

Suppose x and y are nonzero real numbers satisfying the system of equations

$$3x^2 + y^2 = 13x,$$
$$x^2 + 3y^2 = 14y.$$

Find $x + y$.

Problem 5 (CMIMC 2018 Algebra Round P5)

Suppose a, b, and c are nonzero real numbers such that

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a + b + c$.

Problem 6 (PCMM 2018 High School P23)

Let a, b, and c be integers simultaneously satisfying the equations $4abc + a + b + c = 2018$ and $ab + bc + ca = -507$. Find $|a| + |b| + |c|$.

Problem 7 (BMT 2015 Spring Analysis Round P3)

Find all integer solutions to

$$\begin{aligned} x^2 + 2y^2 + 3z^2 &= 36 \\ 3x^2 + 2y^2 + z^2 &= 84 \\ xy + xz + yz &= -7. \end{aligned}$$

Problem 8 (PUMaC 2022 Team Round P7)

Pick x, y, z to be real numbers satisfying $(-x + y + z)^2 - \frac{1}{3} = 4(y - z)^2$, $(x - y + z)^2 - \frac{1}{4} = 4(z - x)^2$, and $(x + y - z)^2 - \frac{1}{5} = 4(x - y)^2$. If the value of $xy + yz + zx$ can be written as $\frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

Problem 9 (SMT 2012 Algebra Individual P8)

For real numbers (x, y, z) satisfying the following equations, find all possible values of $x + y + z$.

$$\begin{aligned} x^2y + y^2z + z^2x &= -1 \\ xy^2 + yz^2 + zx^2 &= 5 \\ xyz &= -2. \end{aligned}$$

Contest Abbreviations

- AMC: American Mathematics Competitions
- AIME: American Invitational Mathematics Examination
- ARML: American Mathematics Regions League
- PCMM: Purple Comet! Math Meet
- CMIMC: Carnegie Mellon Informatics and Mathematics Competition
- SMT: Stanford Math Tournament
- BMT: Berkeley Math Tournament
- PUMaC: Princeton University Mathematics Competition