Tricks for the AIME

Joshua Im

January 18, 2024

This handout is uploaded to my personal website. Visit!

This handout consists of some tricks that work for the AIME. Most of them are bashing, but they will still be helpful.

Keep in mind: AIME is a 3-hour exam with 15 questions. Students have 12 minutes for one problem on average. With the assumption that the first few problems will be done very fast, you'll have at least 30 minutes remaining. If you can get one more question right, then 30 minutes of boring calculation will be worth it.

1 Sequences

For example, suppose you have to find a_{1000} for a sequence $\{a_n\}$. It is humanly impossible to find a_{1000} by hand in 3 hours, right? So one should look for the explicit formula for a_n. This is the best-case scenario, but keep in mind that we only need to get the answer right, regardless of the method.

Conclude with First Few Terms

There should be some sequences that with the first few terms, you can *guess* for the explicit formula. Most of the time, the guessed formula will be correct. This is best explained by an example.

Example 1 (2011 AIME II P11)

Let M_n be the $n \times n$ matrix with entries as follows: for $1 \le i \le n$, $m_{i,i} = 10$; for $1 \le i \le n-1$, $m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n+1}$ can be represented as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

I will first state the right solution.

Solution Using the cofactor expansion theorem twice (I will remove the specific procedure), we get a recursion

$$D_n - 10D_{n-1} + 9D_{n-2} = 0.$$

Therefore, D_n can be represented as

$$D_n = \alpha \cdot 1^n + \beta \cdot 9^n,$$

where α and β are constants. Now, to find α and β, solve the system of equations

$$D_1 = \alpha + 9\beta \quad = 10$$

$$D_2 = \alpha + 81\beta = 91.$$

We get $\alpha = -1/8$, and $\beta = 9/8$. So

$$\frac{1}{8D_n + 1} = \frac{1}{9} \cdot \frac{1}{9^n}.$$

Therefore, the desired value is

$$\frac{1}{9} \sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{9} \cdot \frac{1}{1 - \frac{1}{9}}$$
$$= \frac{1}{72}.$$

Therefore, $p = 1$ and $q = 72$, so $p + q = \boxed{073}$.

Neat, but takes somewhat long, right? Also, in the actual test, we don't know if this solution will come up to your mind. Here is a much faster trick.

Walkthrough 30 second solution

1. Find $\frac{1}{8D_1+1}$.
2. Find $\frac{1}{8D_2+1}$.
3. Find $\frac{1}{8D_3+1}$.
4. Conclude and finish.

Example 2 (2020 AIME II P8)

Define a sequence of functions recursively by $f_1(x) = |x - 1|$ and $f_n(x) = f_{n-1}(|x - n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds 500,000.

Walkthrough Not that simple actually

1. For each $f_n = 0$, what is the mean of the roots?
2. For each $f_n = 0$, how many roots are there?
3. Finish.

Example 3 (2014 AIME II P15)

For any integer $k \geq 1$, let $p(k)$ be the smallest prime that does not divide k. Define the integer function $X(k)$ to be the product of all primes less than $p(k)$ if $p(k) > 2$, and $X(k) = 1$ if $p(k) = 2$. Let $\{x_n\}$ be the sequence defined by $x_0 = 1$, and $x_{n+1}X(x_n) = x_n p(x_n)$ for $n \geq 0$. Find the smallest positive integer, t such that $x_t = 2090$.

Walkthrough This is a hard example.

1. Compute until x_{16} (or x_8).
2. Find that $x_{2^n} = p_n$, where p_n is the nth prime.
3. To have $x_k = p_{n-1} \cdot p_n$, what number should k have?
4. Generally, to have $x_k = p_i \cdot p_n$, what number should k have?
5. Extend this to more than two primes multiplied.
6. Finish.

Cyclic Sequences

If the recursion is given, with asking the nth term, there should be some i such that a_i is very very simple. I mean, VERY simple. One example is stated below.

Example 4 (2012 AIME II P11)

Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Walkthrough The function looks weird. Here's what we can do:

1. Compute $f_2(x)$.
2. Compute $f_3(x)$.
3. Simplify $f_{1001}(x)$.
4. Finish.

2 Counting Bash

When there is a counting problem, people usually use combinatorics tools, but there are some questions that the structure is difficult. For some cases, just pure bashing could be faster. If you have enough time, the probability to get the right idea is far less than the probability to get the right calculation.

Example 5 (2020 AIME II P9)

While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Walkthrough No hints, look for all of them. There aren't that many.

Example 6 (2010 AIME I P10)

Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

Walkthrough There is actually a very clever idea to finish the problem in 20 seconds, but it is hard to find it. To bash, divide cases to $a_3 = 2$, $a_3 = 1$, and $a_3 = 0$.

3 Number Theory Bash

Example 7 (2011 AIME I P11)

Let R be the set of all possible remainders when a number of the form 2^n, n a nonnegative integer, is divided by 1000. Let S be the sum of all elements in R. Find the remainder when S is divided by 1000.

Walkthrough This will actually take 30 minutes. If you don't get the idea. But as I said earlier, it is worth it if you have nothing to do for the rest of the test time.

1. Prove that the remainders are periodic after the first few terms.
2. Prove that the remainders are periodic with period 100.
3. Show that if n is sufficiently large, the remainder of 2^n and the remainder of 2^{n+50} add up to 1000.
4. Finish.

Some of the number theory bashes will come from Diophantine equations. You should mix a decent amount of number theory techniques and calculations. Some computations here are required, so don't get demotivated.

> **Example 8 (2011 AIME I P15)**
> For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

Walkthrough 2011 is the bashing year

1. Get two equations by Vieta.
2. WLOG let $|a| > |b| > |c|$.
3. Show that b and c should have the same sign.
4. Rearrange the first equation and substitute in the second equation.
5. Find the least possible value of a.
6. Try every single case from that a, and finish.

4 Comments

Bashing, at most times, will be helpful. However, some cases don't work. I will give some anti-problems that didn't work. Unfortunately, people wouldn't know if bashing does work or not until they try and waste 20+ minutes. I'm sorry, but I also don't know which problems don't work.

> **Example 9 (2023 AIME I P15)**
> Find the largest prime number $p < 1000$ for which there exists a complex number z satisfying the real and imaginary parts of z are both integers; $|z| = \sqrt{p}$, and there exists a triangle whose three side lengths are p, the real part of z^3, and the imaginary part of z^3.

I took the actual test. I had about 1 hour left and tried this question. If we let $z = a + bi$, then since $p = a^2 + b^2$, p should be a 1 mod 4 prime. I tried all the primes from 997 (since $p < 1000$). My mistake was not realizing

- Checking if a prime works takes too long
- There are a LOT of 1 mod 4 primes below 1000 (about 90)

and I thought the answer would come quickly because $p < 1000$ was given. I spent 40 minutes checking all the primes from 1000 to 509 and gave up. The answer was 349.

Example 10 (2022 AIME II P10)

Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000.

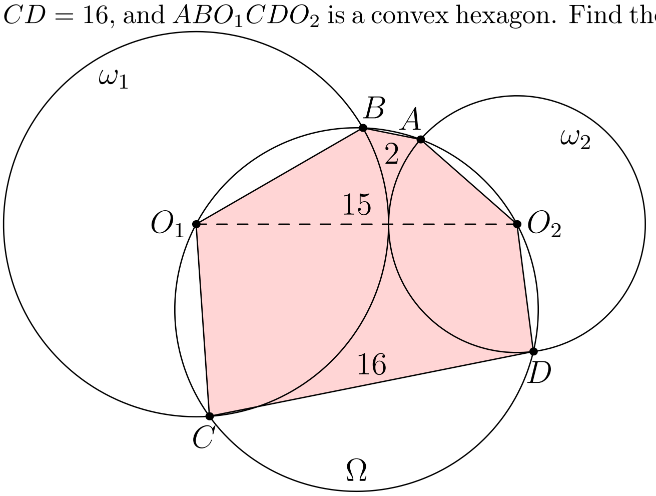
This question should be an easy question with knowing $\sum_{k=1}^{n} k^4$, or at least with knowing how to derive the formula. However, I didn't know, and I tried all the calculation. This involved a lot of multiplications of hundreds, and the answer is wrong even if you compute one number wrong. I should have spent the 30 minutes to derive $\sum_{k=1}^{n} k^4$, not for calculating.

5 Miscellaneous Tricks

Some other techniques that are not in the previous sections are here. Unfortunately, there aren't many examples, so just get to know that it can be guessed like this sometimes. Note that the solutions here include tons of guessing and assuming, so there is no clue that the assumption is correct. But if there's nothing to try, this will be the least worst way to extract the answer.

Example 11 (2022 AIME II P15)

Two externally tangent circles ω_1 and ω_2 have centers O_1 and O_2, respectively. A third circle Ω passing through O_1 and O_2 intersects ω_1 at B and C and ω_2 at A and D, as shown. Suppose that $AB = 2$, $O_1O_2 = 15$, $CD = 16$, and ABO_1CDO_2 is a convex hexagon. Find the area of this hexagon.



Solution We first state the Brahmagupta's formula.

Theorem : Brahmagupta's formula

f a cyclic quadrilateral has side lengths a, b, c, and d, then its area is

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{1}{2}(a+b+c+d)$ is the semiperimeter.

Let r_1 be the radius of circle O_1, and r_2 be the radius of circle O_2. We have $r_1 + r_2 = 15$, so the semiperimeter of the quadrilateral ABO_1O_2 is 16. Then, the area of square ABO_1O_2 is

$$\sqrt{(16-2)(16-r_1)(16-15)(16-r_2)} = \sqrt{14(16-r_1)(16-r_2)}.$$

Since the test asked us for the area, the total area should be an integer. Now assume that the area of quadrilateral ABO_1O_2 has an integer area. In $\sqrt{14(16-r_1)(16-r_2)}$, since 14 is squarefree, we can guess that $(16-r_1)(16-r_2)$ is also a multiple of 14, so that the area is a multiple of 14. The area seems too large for 42 and too small for 14, so we guess that the area is 28.

Then, we have

$$r_1 + r_2 = 15$$
$$(16-r_1)(16-r_2) = 56$$

Solving the system, we get $r_1 = \frac{15+\sqrt{65}}{2}$ and $r_2 = \frac{15-\sqrt{65}}{2}$ (or vise versa).

We now compute the area of the cyclic quadrilateral CDO_2O_1. Since the semiperimeter is 23, the area is

$$\sqrt{(23-16)(23-15)\left(23-\frac{15+\sqrt{65}}{2}\right)\left(23-\frac{15-\sqrt{65}}{2}\right)}$$
$$= \sqrt{7 \cdot 8 \cdot \frac{31-\sqrt{65}}{4} \cdot \frac{31+\sqrt{65}}{2}}$$
$$= 112.$$

Therefore, the total area is $28 + 112 = 140$.

AIME Answers Don't exceed 1000

Example 12 (2022 AIME I P9)

Harold, Tanya, and Ulysses paint a very long picket fence. Harold starts with the first picket and paints every hth picket; Tanya starts with the second picket and paints every tth picket; and Ulysses starts with the third picket and paints every uth picket. Call the positive integer $100h + 10t + u$ *paintable* when the triple (h, t, u) of positive integers results in every picket being painted exactly once. Find the sum of all the paintable integers.

Walkthrough We will find two triplets (h, t, u), and finish,

1. Find one triplet with $h = 3$.
2. Find one triplet with $t = 2$.
3. Prove that $h = 1$ and $h = 2$ don't work.
4. Finish.

Special Angles

Example 13 (2014 AIME II P12)

Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

Solution The maximum degree will be obtuse. WLOG let B be the largest angle.

With the formula

$$\cos 3x = 4\cos^3 x - 3\cos x,$$

we get that the value of $\cos B$ will be either very complicated, or very simple. With hope, we assume that $\cos B$ will be very simple. By the cosine law,

$$m = 10^2 + 13^2 - 2 \cdot 10 \cdot 13 \cdot \cos B.$$

Since m is an integer, $\cos B$ should be rational.

Plugging some values for A, B, and C, we get that $(30°, 120°, 30°)$ works. With not finding any bigger B, we assume that the maximum of m is

$$10^2 + 13^2 - 2 \cdot 10 \cdot 13 \cdot \cos 120° = 399.$$

6 Problems

Sequences

Problem 1 (AMC 12A 2023 P17)

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

Problem 2 (AMC 12A 2023 P25)

There is a unique sequence of integers $a_1, a_2, \cdots a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

Problem 3 (2020 AIME II P6)

Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Problem 4 (2012 AIME II P10)

Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

Problem 5 (2022 AIME II P9)

Let ℓ_A and ℓ_B be two distinct parallel lines. For positive integers m and n, distinct points $A_1, A_2, A_3, \ldots, A_m$ lie on ℓ_A, and distinct points $B_1, B_2, B_3, \ldots, B_n$ lie on ℓ_B. Additionally, when segments $\overline{A_iB_j}$ are drawn for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$, no point strictly between ℓ_A and ℓ_B lies on more than two of the segments. Find the number of bounded regions into which this figure divides the plane when $m = 7$ and $n = 5$.

Problem 6 (2013 AIME II P14)

For positive integers n and k, let $f(n,k)$ be the remainder when n is divided by k, and for $n > 1$ let $F(n) = \max_{1 \le k \le \frac{n}{2}} f(n,k)$. Find the remainder when $\sum_{n=20}^{100} F(n)$ is divided by 1000.

Problem 7 (2000 AIME I P15)

A stack of 2000 cards is labelled with the integers from 1 to 2000, with different integers on different cards. The cards in the stack are not in numerical order. The top card is removed from the stack and placed on the table, and the next card is moved to the bottom of the stack. The new top card is removed from the stack and placed on the table, to the right of the card already there, and the next card in the stack is moved to the bottom of the stack. The process - placing the top card to the right of the cards already on the table and moving the next card in the stack to the bottom of the stack - is repeated until all cards are on the table. It is found that, reading from left to right, the labels on the cards are now in ascending order: $1, 2, 3, \ldots, 1999, 2000$. In the original stack of cards, how many cards were above the card labelled 1999?

Counting Bash

Problem 8 (2010 AIME II P10)

Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

Problem 9 (2022 AIME II P8)

Find the number of positive integers $n \le 600$ whose value can be uniquely determined when the values of $\left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{5} \right\rfloor$, and $\left\lfloor \frac{n}{6} \right\rfloor$ are given, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

Problem 10 (2013 AIME II P9)

A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.

Problem 11 (2016 AIME II P13)

Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labelled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The value of a square is the sum of its row number and column number. The score of an arrangement of rooks is the least value of any occupied square. The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 12 (2017 AIME I P11)

Consider arrangements of the 9 numbers $1, 2, 3, \dots, 9$ in a 3×3 array. For each such arrangement, let a_1, a_2, and a_3 be the medians of the numbers in rows 1, 2, and 3 respectively, and let m be the median of $\{a_1, a_2, a_3\}$. Let Q be the number of arrangements for which $m = 5$. Find the remainder when Q is divided by 1000.

Number Theory Bash

Problem 13 (2018 AIME I P14)

Find the least odd prime factor of $2019^8 + 1$.

Problem 14 (2021 AIME I P10)

Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m+18}{n+19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

7 Answers

Examples

- Example 1 (2011 AIME II P11): 073
- Example 2 (2020 AIME II P8): 101
- Example 3 (2014 AIME II P15): 149
- Example 4 (2012 AIME II P11): 008

- Example 5 (2020 AIME II P9): 090
- Example 6 (2010 AIME I P10): 202
- Example 7 (2011 AIME I P11): 007
- Example 8 (2011 AIME I P15): 098
- Example 9 (2023 AIME I P15): 349
- Example 10 (2022 AIME II P10): 004
- Example 11 (2022 AIME II P15): 140
- Example 12 (2022 AIME I P9): 757
- Example 13 (2014 AIME II P12): 399

Problems

- Problem 1 (AMC 12A 2023 P17): 1/2
- Problem 2 (AMC 12A 2023 P25): -1
- Problem 3 (2020 AIME II P6): 626
- Problem 4 (2012 AIME II P10): 496
- Problem 5 (2022 AIME II P9): 244
- Problem 6 (2013 AIME II P14): 512
- Problem 7 (2000 AIME I P15): 927
- Problem 8 (2010 AIME II P10): 163
- Problem 9 (2022 AIME II P8): 080 or 081
- Problem 10 (2013 AIME II P9): 106
- Problem 11 (2016 AIME II P13): 371
- Problem 12 (2017 AIME I P11): 360
- Problem 13 (2018 AIME I P14): 097
- Problem 14 (2021 AIME I P10): 059