Understanding Differential Equations

For beginners

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Contents

Introduction

This book is about a branch of mathematics that helps us understand how things change over time. Whether you're studying science, engineering, or math, these equations are a key tool to make sense of the world around us.

Imagine you want to describe how a population of animals grows, how an electric circuit behaves, or how a spring moves. Differential equations let us figure out how things change. They are like a magic wand that helps scientists and engineers predict the future based on what we know today.

This book is designed for both mathematicians and engineers. If you're a mathematics enthusiast, this book will provide you with a foundation in differential equations, helping you in further mathematics related to calculus. If you're an aspiring engineer, this book will serve as a practical guide to the theory of differential equations. Engineers use differential equations to solve real-world problems in fields like electrical engineering, mechanical engineering, and civil engineering. Even though there isn't any application example, you'll benefit from the theoretic part of the book. No matter which path you're on, this book will help you to make the most of differential equations in your academic and professional journey.

The book is organized to make learning differential equations easy and accessible as possible. The chapters are:

1. Introduction to Differential Equations: We'll start with the basics, explaining what differential equations are.

2. First-Order Equations: We'll discover how to solve simple differential equations that involve just one unknown function.

3. Higher-Order Equations: We now move on to more complex equations involving derivatives of higher order.

4. Series Solutions: Some differential equations might not have a solution. We so learn how to solve tricky equations using series.

5. Laplace Transforms: We will explore a powerful technique for solving equations differently, making complex problems easier.

6. Systems of Differential Equations: We finally will see how differential equations can describe interactions between multiple things.

To this end, we cover the major topics of ordinary differential equations. These are like the secret codes that help us understand how things change in the world. I hope the reader gets the most from this book, regardless if they're mathematicians or engineers.

I thank Mr. Junwoo Kim for providing a wonderful course on ordinary differential equations in spring 2023 at Korean Minjok Leadership Academy, without him the motivation wouldn't be deep. In addition, I appreciate Yongwook Kim for proofreading and correcting the book to ensure that the manuscript was error-free. I also thank Yebin Song for helping me with the English, Seunghyun Lee for helping me with LaTeX, Minsung Ma for the diagram in section 6.6. Finally, I acknowledge the support and motivation provided by Soeun Kim, Megan Zhang, Jeonghyeon Seo, Dawon Jeong, and everyone who helped me whilst writing this book.

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Chapter 1

Introduction to Differential Equations

Suppose you are modeling something in nature. You find out that the change of the value can be represented by an equation. This is where differential equation starts.

1.1 What is a Differential Equation?

Definition 1.1.1: Differential Equation

A **differential equation** is an equation involving functions' derivatives.

For example, we know that taking the derivative of speed with respect to time equals to velocity. An equation with speed and velocity is one kind of differential equation.

There are lots of types of differential equations, but we first can classify those by types. The two types of differential equations are *ordinary differential equations* and *partial differential equations*. An **ordinary differential equation** is an equation with only derivatives of a single variable. In the other hand, equations including partial derivatives are called **partial differential equations**. This book is focused on ordinary differential equations.

Example 1

Classify these three equations as ordinary differential equations or partial differential equations.

1. $y' = e^x$
2. $\dfrac{dy}{dx} + xy = y^2$
3. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = xy^2$

Solution Equations 1 and 2 are ordinary differential equations because they only contain derivatives of single variables. However, equation 3 is a partial differential equation because z is differentiated by both x and y.

Differential equations can also be classified by order.

Definition 1.1.2: Order of a Differential Equation

An **order** of a differential equation is the order of the highest derivative out of the derivatives in the equation.

For example, the differential equation $\dfrac{d^3y}{dx^3}+y = 1$ has order 3, and the equation $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$ has order 1.

Definition 1.1.3: Linear Differential Equation

A **linear differential equation** is an equation such that the coefficient of $y^{(k)}$ for $k = 1, 2, \ldots$ is independent of y. It can be expressed as

$$a_{(n)}(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0 y = g(x),$$

where $a_0(x)$, $a_1(x)$, $\ldots a_n(x)$ are functions solely dependent on x.

Nonlinear differential equations are simply differential equations that are not linear. If the coefficient of $y^{(k)}$ contains a function of y for some k, then the equation is nonlinear.

Example 2

Classify the following equations as linear and nonlinear.

1. $\sin e^x \dfrac{d^2y}{dx^2} + \cos x \dfrac{dy}{dx} = -xe^x$
2. $yy'' + xy' = x$
3. $x^2y'' + (1 - x^2)y = xy$

Solution Equation 1 is linear because the coefficients of y'', y', and y are functions of x. Equation 2 is nonlinear because the coefficient of the term y'' is y. Equation 3 is linear because the equation can be rewritten as $x^2y'' + (1 - x^2)y' - xy = 0$, which is linear.

Solutions to Differential Equations

Definition 1.1.4: Solution

A function f is a **solution** of a differential equation if it satisfies the differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$.

Solutions may not satisfy the differential equation in all real numbers. Hence, there needs to be clarification in the interval that the solution actually satisfies the differential equation.

Definition 1.1.5: Interval of Definition

The **interval of definition** of a solution is the interval that the solution function satisfies the differential equation.

Example 3

Verify that $y = e^{3x}$ is a solution of the differential equation $y'' - 6y' + 9y = 0$ on the interval $I = (-\infty, \infty)$.

Solution

$$(e^{3x})'' - 6(e^{3x})' + 9(e^{3x}) = 9e^{3x} - 18e^{3x} + 9e^{3x} = 0.$$

In the example above, some might have noticed that $y = 0$ is a solution besides $y = e^{3x}$. Clearly, $e^{3x} \neq 0$. The solution that is zero in all real numbers is called the *trivial solution*.

Definition 1.1.6: Trivial Solution

The **trivial solution** of a differential equation is a solution $f = 0$ for all real numbers that satisfy the differential equation.

Definition 1.1.7: Solution Curve

The **solution curve** of a differential equation is the graph of a solution f.

The solution curve of a differential equation that has order n should be continuous, and should be differentiable at least n times.

Definition 1.1.8: Implicit Solution

A relation $G(x, y) = 0$ is an **implicit solution** if it provides at least one solution to the differential equation.

Definition 1.1.9: Explicit Solution

An **explicit solution** to a differential equation is a solution of the form $y = f(x)$, where f is a function solely dependent on x.

Example 4

Consider the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The relation $x^2 + y^2 = 4$ is an implicit solution because $y = -\sqrt{4 - x^2}$ is a solution to the differential equation, and it also satisfies the relation $x^2+y^2 = 4$.

Family of Solutions

Notice that from the example above, $x^2 + y^2 = c$ can be an implicit solution for an arbitrary positive constant c. This shows that there could be infinitely many solutions to differential equations. The set of solutions containing a constant is called a *family of solutions*.

Definition 1.1.10: Family of Solutions

For a constant c, the relation $G(x, y, c) = 0$ is called a **one-parameter family of solutions**. For constants c_1, c_2, ..., c_n, the relation $G(x, y, c_1, c_2, \ldots, c_n) = 0$ is called a **n-parameter family of solutions**.

Definition 1.1.11: Particular Solution

A solution of a differential equation that doesn't contain any parameters is called a **particular solution**.

Example 5

Verify that $y = c_1x^4 + c_2x^{-1}$ is a solution to the differential equation

$$x^2y'' - 2xy' - 4y = 0,$$

where c_1 and c_2 are parameters.

Solution

$$\begin{aligned}
&x^2\frac{d^2y}{dx^2}(c_1x^4 + c_2x^{-1}) - 2x\frac{dy}{dx}(c_1x^4 + c_2x^{-1}) - 4\frac{d^2y}{dx^2}(c_1x^4 + c_2x^{-1}) \\
&= c_1(12x^4 - 8x^4 - 4x^4) + c_2(2x^{-1} + 2x^{-1} - 4x^{-1}) \\
&= 0.
\end{aligned}$$

Usually, all solutions of a differential equation will be in the family of solutions, but there may be some solutions that are not in the family of the solutions. For example, the differential equation $xy' = y$ has $y = cx^2$ as a solution, but consider the solution

$$y(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0. \end{cases}$$

This solution can not be expressed as $y = cx^2$. Such solutions are called *singular solutions.*

Definition 1.1.12: Singular Solution

An extra solution that is not in the family of solution is called a **singular solution**.

1.2 Initial-Value Problems

Definition 1.2.1: Initial-Value Problems

A differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

with initial conditions

$$y(x_0) = c_0,\ y'(x_0) = c_1,\ \ldots,\ y^{(n-1)}(x_0) = c_{n-1}$$

is called an **initial-value problem**.

For example, the equation

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = c_0$$

is a 1st-order initial-value problem.

Example 1

Recall from Example 5 from section 1.1 that $y = c_1 x^4 + c_2 x^{-1}$ is a solution to the differential equation

$$x^2 y'' - 2xy' - 4y = 0.$$

Solve the initial-value problem

$$x^2 y'' - 2xy' - 4y = 0,\ y(1) = 3,\ y'(1) = 2.$$

Solution Since $y(1) = 3$, $c_1 + c_2 = 3$. Differentiating $y = c_1 x^4 + c_2 x^{-1}$, we get $y' = 4c_1 x^3 - c_2 x^{-2}$. Therefore, $y'(1) = 4c_1 - c_2 = 2$. Solving the system of linear equations

$$c_1 + c_2 = 3$$

$$4c_1 - c_2 = 2$$

gives $c_1 = 1$, $c_2 = 2$, and hence the solution to the initial value problem is

$$y(x) = x^4 + 2x^{-1}.$$

Existence and Uniqueness

For initial-value problems, two fundamental questions arise: *existence* and *uniqueness*. Does the solution exist, and if there exists a solution, is it unique? The following theorem doesn't totally clarify the question but gives you an idea of it.

Theorem 1.2.1: Picard–Lindelöf theorem

Let R be a rectangular region defined by $R = [a, b] \times [c, d]$ that contains (x_0, y_0). If $f(x, y)$ is continuous on R and has bounded first partial derivative with respect to y on R, that is, if $\partial f/\partial y$ is bounded on $a \leq x \leq b$, then the initial-value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution on the interval $(x_0 - \epsilon, x_0 + \epsilon)$.

This theorem guarantees a unique solution locally, but the condition is not accessible. We state a similar theorem which is more useful.

Theorem 1.2.2: Existence and Uniqueness Theorem

Let R be a rectangular region defined by $R = [a, b] \times [c, d]$ that contains (x_0, y_0). If $f(x, y)$ and $\partial f/\partial y$ is continuous on R, then the initial-value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution on the interval $(x_0 - \epsilon, x_0 + \epsilon)$.

Example 2

Prove that $y = x^6$ is a unique solution to the initial-value problem

$$\frac{dy}{dx} = 6xy^{2/3}, \qquad y(1) = 1$$

near $x = 1$.

Solution First, $y = x^6$ is a solution to the differential equation $dy/dx = 6xy^{2/3}$ because

$$\frac{dy}{dx} = 6x^5 = 6x \cdot x^{6 \cdot 2/3}.$$

Since $f(x, y) = 6xy^{2/3}$ and $\partial f/\partial y = 4xy^{-1/3}$ is continuous near $x = 1$, $y = x^6$ is the unique solution to the initial-value problem.

Notice that the converse may not be true: that is, having a unique solution near (x_0, y_0) doesn't guarantee that $f(x, y)$ and $\partial f/\partial y$ is continuous near (x_0, y_0). Note that this theorem only guarantees a unique solution near x_0. This means that the solution to the initial-value problem may not have a unique solution globally. One example is stated below.

Example 3

Verify that

$$y(x) = \begin{cases} 0 & x < 0 \\ x^6 & x \geq 0 \end{cases}$$

is a solution to the initial-value problem

$$\frac{dy}{dx} = 6xy^{2/3}, \qquad y(1) = 1.$$

Solution For $x \geq 0$, we have

$$\frac{dy}{dx} = 6x^5 = 6x \cdot x^{6 \cdot 2/3}.$$

For $x < 0$, we have $dy/dx = 6xy^{2/3} = 0$. Therefore, $y(x)$ is a solution to the initial-value problem.

The solution given in example 3 is clearly different from the solution given in example 2. They are the same near $x = 1$, but different globally. Keep in mind that the existence and uniqueness theorem only guarantees a unique solution locally.

1.3 Direction Fields

Consider a first-order equation

$$\frac{dy}{dx} = f(x, y).$$

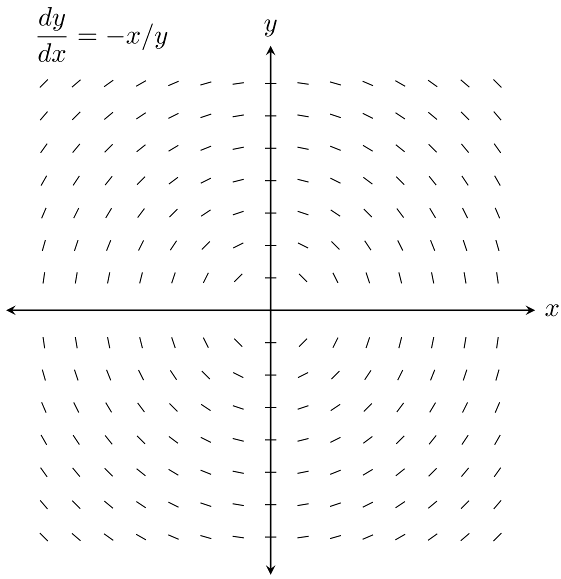
Then, at a point (x_0, y_0), $f(x_0, y_0)$ can be interpreted as the slope of the tangent line of the solution at (x_0, y_0). Since the value of $f(x, y)$ changes, the slope of the tangent line of the solution will change as (x_0, y_0) changes. The diagram where all the slopes are drawn for each point is called a *direction field.*

Example 1

Consider a first-order differential equation

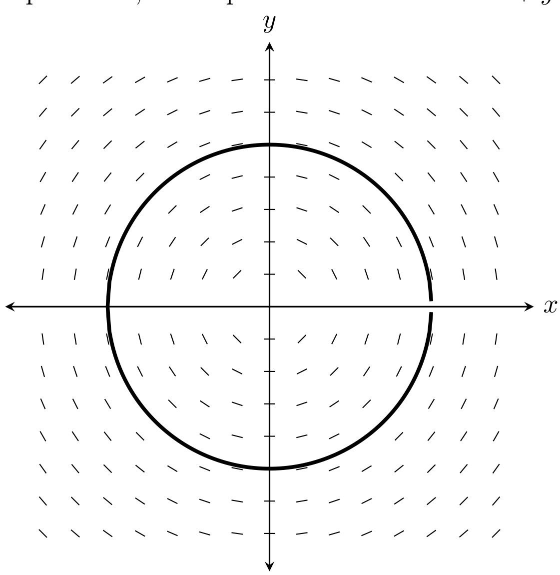
$$\frac{dy}{dx} = -\frac{x}{y}.$$

For each (x_0, y_0), the slope of the tangent line of the solution is equal to $-x_0/y_0$. The direction field is drawn below.



Note that the direction field is not drawn for $y = 0$ since it is impossible to divide by zero.

With direction fields, we can find how the solution behaves. For the example above, it seems like the slopes form circles. This is because that $x^2 + y^2 = c$ is a solution for a constant c. One can guess the form of the family of solutions with the direction field. If there is an initial condition, then the solution can be approximated. For example, if there was an initial condition $y(4) = 3$ to the example above, the implicit solution will be $x^2 + y^2 = 25$.



See that the solution curve exactly fits with some elements in the direction field.

Chapter 2

First-Order Equations

This chapter covers about first-order differential equations. There are several kinds of first-order differential equations that can be solved analytically.

2.1 Separable Equations

Definition 2.1.1: Separable Equation

A first-order differential equation is **separable** if it can be expressed as

$$\frac{dy}{dx} = g(x)h(y).$$

The equation is called separable because $dy/dx = f(x, y)$ can be separated as a multiplication of two functions, one depending only on x, and the other depending only on y.

Solving Separable Equations

The method for solving separable equations is not difficult. First, change

$$\frac{dy}{dx} = g(x)h(y)$$

to

$$\frac{1}{h(y)}\,dy = g(x)\,dx.$$

Then, the left-hand side is a function solely depending on y, and the right-hand side is a function solely depending on x. Integrating both sides gives

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx \text{ and}$$

$$H(y) + c_1 = G(x) + c_2$$

where $H(y)$ and $G(x)$ are antiderivatives of $1/h(y)$ and $g(x)$, respectively. Subtracting c_1 in both sides gives you $H(y) = G(x) + c$, where $c = c_2 - c_1$.

Example 1

Solve $y\,dx + (1 + x^2)\,dy = 0$.

Solution Since $dy/dx = -y/(1 + x^2) = -y \cdot 1/(1 + x^2)$, the equation is separable. Rewriting the equation gives

$$\frac{1}{y}\,dy = -\frac{1}{1+x^2}\,dx,$$

and by integrating both sides, we get

$$\ln|y| = -\tan^{-1}x + c_1.$$

Therefore,

$$\begin{aligned}|y| &= e^{-\tan^{-1}x + c_1} \\ &= e^{c_1}\cdot e^{-\tan^{-1}x} \\ &= ce^{-\tan^{-1}x}\end{aligned}$$

where $e^{c_1} = c$.

Example 2

Solve $\frac{1}{3}x\,dx + \frac{1}{3}y\,dy = 0$.

Solution Since $dy/dx = -x/y$, the equation is separable. Rewriting the equation gives

$$\begin{aligned}x\,dx &= -y\,dy \\ \int x\,dx &= -\int y\,dy \\ \frac{1}{2}x^2 &= -\frac{1}{2}y^2 + c_1\end{aligned}$$

and therefore we get the implicit solution $x^2 + y^2 = c$ where $2c_1 = c$. The explicit solution is

$$y = \pm\sqrt{c - x^2}.$$

Notice that x can only be defined where $c - x^2 \geq 0$, so the interval of definition is $(-\sqrt{c}, \sqrt{c})$. x is not defined at $-\sqrt{c}$ or $\sqrt{c}$ because $y(x)$ should be differentiable.

Example 3

Solve $(1 + x^3)\,dy - x^2y\,dx = 0$, $y(0) = 2$.

Solution The equation is separable. Rewriting the equation gives

$$\begin{aligned}\frac{x^2}{1 + x^3}\,dx &= \frac{1}{y}\,dy \\ \int \frac{x^2}{1 + x^3}\,dx &= \int \frac{1}{y}\,dy \\ \frac{1}{3}\ln|1 + x^3| + c_1 &= \ln|y|.\end{aligned}$$

Then,

$$\begin{aligned} |y| &= e^{\frac{1}{3}\ln|1+x^3|+c_1} \\ &= c_2 \cdot e^{\frac{1}{3}\ln|1+x^3|} \\ &= c_2 \cdot |1+x^3|^{1/3}, \end{aligned}$$

and $y = c|1+x^3|^{1/3}$, where $c_2 = e^{c_1}$, and $c = \pm c_2$. Since $y(0) = 2$, $c \cdot |1+0^3|^{1/3} = c = 2$. Therefore, we get the solution

$$y = 2|1+x^3|^{1/3}.$$

Homogeneous Equations of the Same Degree

Definition 2.1.2: Homogeneous Function of Degree n

If a function f satisfies the property

$$f(tx, ty) = t^n f(x, y),$$

we say that f is **homogeneous of degree n**.

For example, $f(x, y) = xy$ is homogeneous of degree 2 because $f(tx, ty) = t^2 xy$. This term *homogeneous* is different with the term where the constant function is zero. Homogeneous equations can be solved by substitution.

Definition 2.1.3: Homogeneous Equation of the Same Degree

A first order differential equation is **homogeneous of the same degree** where it is of the form

$$f(x, y)\, dx + g(x, y)\, dy = 0$$

and $f(x, y)$ and $g(x, y)$ are homogeneous of the same degree.

Solving Homogeneous Equations of the Same Degree

Let $x/y = u$ so $x = yu$. Then, $f(x, y) = f(yu, y) = y^n f(u, 1)$, and $g(x, y) = y^n g(u, 1)$. Since $x = yu$, $dx = y\,du + u\,dy$. This gives

$$\begin{aligned} y^n f(u,1)\,dx + y^n g(u,1)\,dy &= 0 \\ f(u,1)\,dx + g(u,1)\,dy &= 0 \\ f(u,1)\,(y\,du + u\,dy) + g(u,1)\,dy &= 0 \\ yf(u,1)\,du + \big(uf(u,1) + g(u,1)\big)\,dy &= 0 \\ \frac{f(u,1)}{uf(u,1)+g(u,1)}\,du &= -\frac{1}{y}\,dy, \end{aligned}$$

which is a separable equation of y and u. The procedure after this is the same with other separable equations. This process can also be done by the substitution $y/x = v$, or $y = xv$. Either way will result in a separable equation, so there is no need to worry which substitution you should make. Go for the one that looks simpler.

Example 4

Solve $(x^3 - y^3)\,dx + xy^2\,dy = 0$.

Solution Since $x^3 - y^3$ and xy^2 are homogeneous of the same degree, we use the substitution $x = yu$ to change the equation to a separable equation.

$$\begin{aligned} y^3(u^3-1)\,dx + y^3 u\,dy &= 0 \\ (u^3-1)\,dx + u\,dy &= 0 \\ (u^3-1)(y\,du + u\,dy) + u\,dy &= 0 \\ (u^3-1)y\,du + u^4\,dy &= 0 \\ \frac{u^3-1}{u^4}\,du &= -\frac{1}{y}\,dy. \end{aligned}$$

Integrating both side leads to the solution

$$\begin{aligned}
\int \frac{u^3-1}{u^4}\,du &= -\int \frac{1}{y}\,dy \\
\ln u + \frac{1}{3}u^{-3} + &= -\ln y + c_1 \\
\ln \frac{x}{y} + \frac{y^3}{3x^3} &= -\ln y + c_1 \\
\ln x + \frac{y^3}{3x^3} &= c_1 \\
y^3 &= x^3(c - 3\ln x) \\
y &= x\sqrt[3]{c - 3\ln x},
\end{aligned}$$

where $c = 3c_1$.

Reduction to Separable Form

There are some other substitutions that change the equation into a separable form. Consider the equation

$$\frac{dy}{dx} = f(ax + by + c),$$

where a, b, and c are constants. Substituting $u = ax + by + c$ gives

$$\frac{dy}{dx} = f(u).$$

Since

$$\begin{aligned}
\frac{du}{dx} &= a + b\frac{dy}{dx}, \\
\frac{dy}{dx} &= \frac{1}{b}\left(\frac{du}{dx} - a\right)
\end{aligned}$$

and the equation can be changed into

$$\begin{aligned}
\frac{1}{b}\left(\frac{du}{dx} - a\right) &= f(u) \\
\frac{du}{dx} &= a + bf(u),
\end{aligned}$$

which is separable, and can be solved by our usual method of solving separable equations.

Example 5

Solve $\dfrac{dy}{dx} = (2x + 3y - 6)^2 + \dfrac{1}{3}$.

Solution Let $u = 2x + 3y - 6$. Then,

$$\begin{aligned} \frac{du}{dx} &= 2 + 3\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{3}\left(\frac{du}{dx} - 2\right), \end{aligned}$$

so the equation can be changed into

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = u^2 + \frac{1}{3},$$

which is

$$\begin{aligned} \frac{du}{dx} &= 3u^2 + 3 \\ \frac{1}{u^2+1}\, du &= 3\, dx. \end{aligned}$$

So the equation is separable. Integrating,

$$\begin{aligned} \int \frac{1}{u^2+1}\, du &= 3 \int dx \\ \arctan u &= 3x \end{aligned}$$

Substituting $u = 2x + 3y - 6$ back to the equation yields

$$\begin{aligned} \arctan(2x + 3y - 6) &= 3x + c \\ 2x + 3y - 6 &= \tan(3x + c), \end{aligned}$$

Therefore the solution is

$$y = \frac{1}{3}(-2x + \tan(3x + c) + 6).$$

2.2 Exact Equations

Recall the definition of total differential. If $z(x, y)$ is a function with two variables such that it has continuous partial derivatives, its total differential is defined as

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

If $z(x, y) = c$, where c is a constant, then $dz = 0$. This is where exact equations start.

Definition 2.2.1: Exact Equation

A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called to be **exact** if the left-hand side is the total differential of some function $z(x, y)$. That is, if there exists a function $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y).$$

Then, how do you know if a first-order differential equation is exact? The theorem below answers to the question.

Theorem 2.2.1: Determining Exact Equations

Consider a first-order differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

where $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives. The equation is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof. ($\Rightarrow$) Since the equation is exact, there exists a function $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y).$$

Since $M(x, y)$ and $N(x, y)$ have continuous partial derivatives, $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 z}{\partial x \partial y}$ is continuous, and $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 z}{\partial y \partial x}$ is continuous. Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

because

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

by Clairaut's Theorem.

($\Leftarrow$) We claim that there exists a function $z(x, y)$ if $\partial M/\partial y = \partial N/\partial x$. Such function should have $M(x, y)$ as its first partial derivative with respect to x. Therefore,

$$z(x, y) = \int M(x, y)\, dx + g(y),$$

where $g(y)$ is an arbitrary function of y. Therefore, we can guarantee that there exists a function z if there exists a function $g(y)$ which is independent to x. Since $z(x, y)$ should have $N(x, y)$ as its first partial derivative with respect to y,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y) = N(x, y), \text{ and}$$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx.$$

We now claim that $g(y)$ is a function independent to x. Taking partial derivatives with respect to x,

$$\begin{aligned} \frac{\partial}{\partial x}\left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx\right) &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} \int M(x, y)\, dx\right) \\ &= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \int M(x, y)\, dx\right) \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0. \end{aligned}$$

So there exists a function $g(y)$ that is independent to x. Therefore, there exists a function $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y). \qquad \blacksquare$$

Solving Exact Equations

The goal of solving an exact equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is to find a function $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y)$$

so that we can conclude that the solution is $z(x, y) = c$, where c is a constant. Since $\partial z/\partial x = M(x, y)$,

$$z(x, y) = \int M(x, y)\, dx + g(y),$$

where $g(y)$ is a solution solely dependent on y. Taking the partial derivative with y gives

$$\frac{\partial}{\partial y} \int M(x, y)\, dx + g'(y) = N(x, y),$$

which gives the formula of $g'(y)$, with knowing $\int M(x, y)$ and $N(x, y)$. Taking the antiderivative gives $g(y)$, and one can find $z(x, y)$. The method can also be done the other way, starting with integrating with respect to y first. The solution to the differential equation is

$$z(x, y) = c.$$

Example 1

Solve $e^x \sin e^x y^2\, dx - 2 \cos e^x y = 0$.

Solution In this equation, $M(x, y) = e^x \sin e^x y^2$, and $N(x, y) = -2 \cos e^x y$. Since

$$\frac{\partial M}{\partial y} = 2e^x \sin e^x = \frac{\partial N}{\partial x},$$

the equation is exact. Therefore, we need to find $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y).$$

Integrating $N(x, y) = -2 \cos e^x y$ with respect to y gives

$$\begin{aligned} z(x, y) &= \int N(x, y)\, dy + f(x) \\ &= -2 \int \cos e^x y\, dy + f(x) \\ &= -\cos e^x y^2 + f(x). \end{aligned}$$

Since $\partial z/\partial x = M(x, y)$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\Big(-\cos e^x y^2 + f(x)\Big)\\ &= e^x \sin e^x y^2 + f'(x) = e^x \sin e^x y^2,\end{aligned}$$

which gives $f'(x) = 0$, and $f(x) = c_1$. Therefore, the solution is $z(x, y) = c$, which is

$$-\cos e^x y^2 = c.$$

Example 2

Solve $(x^2 - 2xy)\,dx + (y^3 - x^2)\,dy = 0$, $y(0) = 2$.

Solution In this equation, $M(x, y) = x^2 - 2xy$, and $N(x, y) = y^3 - x^2$. Since

$$\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x},$$

the equation is exact. Therefore, we need to find $z(x, y)$ such that

$$\frac{\partial z}{\partial x} = M(x, y) \text{ and } \frac{\partial z}{\partial y} = N(x, y).$$

Integrating $M(x, y) = x^2 - 2xy$ with respect to x gives

$$\begin{aligned}z(x, y) &= \int N(x, y)\,dx + g(y)\\ &= \int (x^2 - 2xy)\,dx + g(y)\\ &= \frac{1}{3}x^3 - x^2 y + g(y).\end{aligned}$$

Since $\partial z/\partial y = N(x, y)$,

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\Big(\frac{1}{3}x^3 - x^2 y + g(y)\Big)\\ &= -x^2 + g'(y) = y^3 - x^2,\end{aligned}$$

which gives $g'(y) = y^3$, and $g(y) = \frac{1}{4}y^4 + c_1$. Therefore, the solution is $z(x, y) = c$, which is

$$\frac{1}{3}x^3 - x^2 y + \frac{1}{4}y^4 = c.$$

With the initial condition $y(0) = 2$, we get $c = 4$, so the solution of the equation is

$$\frac{1}{3}x^3 - x^2 y + \frac{1}{4}y^4 = 4.$$

Reduction to Exact Form

There are some equations that look like exact, but actually they're not. For some cases, nonexact equations can be changed to exact equations by multiplying some function to both sides of the equations! For example, the equation

$$4y^2\,dx + 2xy\,dy = 0$$

is not exact because $\partial M/\partial y = 8y$, and $\partial N/\partial x = 2y$, so $\partial M/\partial y \neq \partial N/\partial x$. However, multiplying the integrating factor x^3 gives

$$4x^3y^2\,dx + 2x^4y\,dy = 0,$$

and this is exact because $\partial M/\partial y = 8x^3y = \partial N/\partial x$. Then, how do we find the integrating factor? Suppose there exists an integrating factor $u(x, y)$ which makes

$$u(x, y)M(x, y)\,dx + u(x, y)N(x, y)\,dy = 0$$

to an exact equation. Then, by theorem 2.2.1,

$$\begin{aligned}\frac{\partial}{\partial y}u(x, y)M(x, y) &= \frac{\partial}{\partial x}u(x, y)N(x, y)\\ u_yM + uM_y &= u_xN + uN_x\\ (M_y - N_x)u &= u_xN - u_yM\end{aligned}$$

Assume that u is a function of only one variable. That is, u only depends on either x or y. If u only depends on x, then since $u_y = 0$ and $u_x = du/dx$,

$$\begin{aligned}(M_y - N_x)u &= u_xN\\ \frac{du}{dx} &= \frac{M_y - N_x}{N}u.\end{aligned}$$

Notice that $(M_y - N_x)/N$ is independent u, hence the equation above is separable. Solving the equation for u gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{M_y - N_x}{N}\,dx\\ \ln|u| &= \int \frac{M_y - N_x}{N}\,dx\\ u &= e^{\int (M_y - N_x)/N\,dx}.\end{aligned}$$

Since the right-hand side of the original equation is 0, the sign of u doesn't matter, and absolute value can be ignored. If u only depends on y, then

$$\begin{aligned}(M_y - N_x)u &= -u_y M \\ \frac{du}{dy} &= \frac{N_x - M_y}{M} u.\end{aligned}$$

Since the equation is separable,

$$\begin{aligned}\frac{1}{u}\,du &= \frac{N_x - M_y}{M}\,dy \\ \ln|u| &= \int \frac{N_x - M_y}{M}\,dy \\ u &= e^{\int (N_x - M_y)/M\,dy}.\end{aligned}$$

Therefore, nonexact first-order differential equations can be changed to exact equations if $(M_y - N_x)/N$ depends only on x, or if $(N_x - M_y)/M$ depends only on y. One should check if the equation is exact first, before looking for the integrating factor.

Example 3

Solve $(x^2 + y^2)\,dx + xy\,dy = 0$.

Solution Let $M(x, y) = x^2 + y^2$, and $N(x, y) = xy$. The equation is not exact because $M_y = 2y$ and $N_x = y$, and $M_y \neq N_x$. However, since

$$\frac{M_y - N_x}{N} = \frac{2y - y}{xy} = \frac{1}{x}$$

which is a function only depending on x, there exists an integrating factor

$$u(x, y) = e^{\int (1/x) dx} = x.$$

Multiplying x to the equation gives

$$(x^3 + xy^2)\,dx + x^2 y\,dy = 0,$$

which is exact. Solving for $z(x, y)$,

$$\begin{aligned}z(x, y) &= \int (x^2 y)\,dy + f(x) \\ &= \frac{1}{2}x^2 y^2 + f(x).\end{aligned}$$

Since $\partial z/\partial x = x^3 + xy^2$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}x^2y^2 + f(x)\right)\\ &= xy^2 + f'(x) = x^3 + xy^2,\end{aligned}$$

which gives $f'(x) = x^3$, and $f(x) = \frac{1}{4}x^4 + c_1$. Therefore, the solution is $z(x, y) = c$, which is

$$\frac{1}{2}x^2y^2 + \frac{1}{4}x^4 = c.$$

2.3 Linear Equations

Definition 2.3.1: Linear Equation

A first-order differential equation of the form

$$p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called to be **linear**.

First-order differential equations that cannot be expressed in this form are called *nonlinear equations*. It is also called *homogeneous* if $r(x) = 0$ in the formula above. There are two ways to solve linear equations. Both are stated.

Variation of Parameter Method

First, change the coefficient of dy/dx to 1. This gives

$$\frac{dy}{dx} + f(x)y = g(x)$$

where $f(x) = q(x)/p(x)$ and $g(x) = r(x)/p(x)$. The idea is to find a solution to the differential equation

$$\frac{dy}{dx} + f(x)y = 0,$$

which is called the complementary solution, and a solution to the differential equation

$$\frac{dy}{dx} + f(x)y = g(x),$$

which is called the particular solution. They are denoted y_c and y_p, respectively. This will be explained later, in section 3.1. Then, the general solution is $y_c + y_p$

because

$$\frac{d}{dx}(y_c + y_p) + f(x)(y_c + y_p) = \left(\frac{d}{dx}y_c + f(x)y_c\right) + \left(\frac{d}{dx}y_p + f(x)y_p\right)$$
$$= 0 + g(x) = g(x).$$

To find the complementary solution, we need to find the solution of the equation

$$\frac{dy}{dx} + f(x)y = 0.$$

Notice that the equation above is separable. Solving for y gives

$$\begin{aligned}
\frac{dy}{dx} &= -f(x)y \\
\frac{1}{y}\,dy &= -f(x)\,dx \\
\ln|y| &= -\int f(x)\,dx + c \\
y &= ce^{-\int f(x)\,dx},
\end{aligned}$$

which is the complementary solution. Let $e^{-\int f(x)\,dx} = y'(x)$.

For the particular solution, we will use the variation of parameter method, which is a process for finding $u(x)$ where we assume $y_p(x) = u(x)y'(x)$. This is explained later in section 3.6. Substituting y_p into the equation,

$$\begin{aligned}
\frac{d}{dx}y_p + f(x)y_p &= g(x) \\
\frac{d}{dx}(uy') + f(x)(uy') &= g(x) \\
u\frac{d}{dx}y' + y'\frac{d}{dx}u + f(x)uy' &= g(x) \\
u\left(\frac{d}{dx}y' + f(x)y'\right) + y'\frac{d}{dx}u &= g(x)
\end{aligned}$$

The first part of the left-hand side is 0 since y' is a complementary solution. Therefore, we get

$$y'\frac{du}{dx} = g(x),$$

which is separable. Solving the separable equation gives

$$
\begin{aligned}
du &= \frac{g(x)}{y'}\,dx \\
\int du &= \int \frac{g(x)}{y'}\,dx \\
u(x) &= \int \frac{g(x)}{y'}\,dx.
\end{aligned}
$$

Therefore, the general solution to the linear equation is

$$
\begin{aligned}
y &= y_c + y_p \\
&= ce^{-\int f(x)\,dx} + e^{-\int f(x)\,dx} \int g(x) e^{\int f(x)\,dx}\,dx.
\end{aligned}
$$

Integrating Factor Method

The idea of the integrating factor method is this: we want to find an integrating factor $u(x)$ so that the whole equation multiplied by the integrating factor is the derivative of $u(x)y$. To find such $u(x)$, first multiply the whole equation by $u(x)$, changing the equation to

$$
u(x)\frac{dy}{dx} + f(x)u(x)y = u(x)g(x).
$$

We want the left-hand side to be the derivative of $u(x)y$. That is,

$$
\begin{aligned}
\frac{d}{dx}(u(x)y) &= y\frac{du}{dx} + u(x)\frac{dy}{dx} \\
&= u(x)\frac{dy}{dx} + f(x)u(x)y,
\end{aligned}
$$

which gives a separable equation

$$
\frac{du}{dx} = f(x)u(x).
$$

Solving for u gives

$$\begin{aligned} \frac{1}{u}\,du &= f(x)\,dx \\ \int \frac{1}{u}\,du &= \int f(x)\,dx \\ \ln|u| &= e^{\int f(x)\,dx} \\ u(x) &= c_1 e^{\int f(x)\,dx}. \end{aligned}$$

We only need one integrating factor, so we fix $c_1 = 1$ to make calculations simple. To find the solution, since

$$\frac{d}{dx}(u(x)y) = u(x)\frac{dy}{dx} + f(x)u(x)y = u(x)g(x),$$

we can find $u(x)y$ by taking the integral.

$$u(x)y = \int u(x)g(x)\,dx + c$$

Thus, the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)}\left(\int u(x)g(x) + c\right) \text{ where} \\ u(x) &= e^{\int f(x)\,dx}. \end{aligned}$$

Example 1

Solve $\dfrac{dy}{dx} + y = e^{3x}$.

Solution The differential equation is linear, where $f(x) = 1$ and $g(x) = e^{3x}$. The integrating factor is

$$\begin{aligned} u(x) &= e^{\int f(x)\,dx} \\ &= e^{\int 1\,dx} = e^x. \end{aligned}$$

Hence, the solution is

$$\begin{aligned} y &= e^{-x}\left(\int e^x \cdot e^{3x} + c\right) \\ &= e^{-x}\int e^{4x}\,dx + ce^{-x} \\ &= e^{-x}\cdot\frac{1}{4}e^{4x} + ce^{-x} \\ &= \frac{1}{4}e^{3x} + ce^{-x}. \end{aligned}$$

Example 2

Solve $x\dfrac{dy}{dx} + 4y = x^3 - x,\ y(1) = -\dfrac{37}{35}$.

Solution The differential equation is linear because dividing x to both sides gives

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1.$$

Notice that $f(x) = 4/x$ is continuous on intervals $(-\infty, 0)$ and $(0, \infty)$. Since the initial value is at 1, we solve the equation in the interval $(0, \infty)$. The integrating factor is

$$\begin{aligned} u(x) &= e^{\int f(x)\,dx} \\ &= e^{\int (4/x)\,dx} \\ &= e^{4\ln|x|} = x^4. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= x^{-4}\left(\int x^4(x^2-1) + c\right) \\ &= x^{-4}\int (x^6 - x^4)\,dx + cx^{-4} \\ &= x^{-4}\left(\frac{1}{7}x^7 - \frac{1}{5}x^5\right) + cx^{-4} \\ &= \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}. \end{aligned}$$

With the initial value $y(0) = -37/35$, we get $c = -1$. Hence, the solution for the equation in the interval $(0, \infty)$ is

$$y = \frac{1}{7}x^3 - \frac{1}{5}x - x^{-4}.$$

Reduction to Linear Form

Some nonlinear equations can be reduced to linear form by substituting. One of them are called the *Bernoulli's equation.*

Definition 2.3.2: Bernoulli's Equation

A first-order nonlinear differential equation of the form

$$\frac{dy}{dx} + f(x)y = g(x)y^{\alpha},$$

where α is any real number, is called **Bernoulli's equation**.

Let $u = y^{1-\alpha}$. Then $du/dx = (1-\alpha)y^{-\alpha}dy/dx$. Substituting this to Bernoulli's Equation gives

$$\frac{1}{1-\alpha} \cdot y^{\alpha}\frac{du}{dx} + f(x)y = g(x)y^{\alpha}.$$

Dividing both sides by y^{α}, the equation is changed into

$$\frac{1}{1-\alpha}\frac{du}{dx} + f(x)y^{1-\alpha} = g(x),$$

which is linear because it is equivalent to

$$\frac{du}{dx} + (1-\alpha)f(x)u = (1-\alpha)g(x).$$

Example 3

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^3$.

Solution The equation is Bernoulli's Equation, where $\alpha = 3$. We make the substitution $u = y^{-2}$. Then, $du/dx = -2y^{-3}dy/dx$. Substituting to the original equation gives

$$-\frac{1}{2}y^3 \cdot \frac{du}{dx} + \frac{y}{x} = x^2y^3$$

$$-\frac{1}{2}\frac{du}{dx} + \frac{1}{x}y^{-2} = x^2$$

$$\frac{du}{dx} - \frac{2}{x}u = -2x^2.$$

The integrating factor is

$$\begin{aligned}\mu(x) &= e^{\int f(x)\,dx}\\ &= e^{-\int (2/x)\,dx}\\ &= e^{-2\ln|x|} = x^{-2}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}u &= x^2\left(\int x^{-2}(-2x^2) + c\right)\\ &= -2x^2\int dx + cx^2\\ &= -2x^3 + cx^2.\end{aligned}$$

Since $u = y^{-2} = -2x^3 + cx^2$, solving for y gives

$$\begin{aligned}y &= \frac{1}{\sqrt{-2x^3 + cx^2}}\\ &= \frac{1}{x\sqrt{-2x + c}}.\end{aligned}$$

Chapter 3

Higher-Order Equations

This chapter mainly covers linear and nonlinear equations with higher orders.

3.1 Theory of the General Solution

Recall the n-th order linear initial-value problem

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

$$y(x_0) = y_0,\, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_{n-1}) = y_{n-1}.$$

This chapter, in general, focuses on this form of equations. Recall that there was a theorem about the existence and uniqueness of solutions in section 1.2. There is a similar theorem, stating the existence and uniqueness of a solution.

Theorem 3.1.1: Existence and Uniqueness of a Solution

Let $a_n(x)$, $a_{n-1}(x)$, …, $a_0(x)$ be continuous functions on an interval I, and let $a_n(x) \neq 0$, and $x_0 \in I$. Then, the solution to the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

$$y(x_0) = y_0,\, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_{n-1}) = y_{n-1}$$

exists, and it is unique.

Solving linear differential equations is divided into two parts-the homogeneous equation, and the nonhomogeneous equation.

Homogeneous Equations

Recall that an equation is homogeneous if $f(x) = 0$. That is, if a differential equation is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Homogeneous equations usually have solutions with parameters if they do not have initial conditions.

Theorem 3.1.2: Superposition Principle - Homogeneous Equations

Let y_1, y_2, …, y_k be the solutions to the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Then, the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k$$

is a solution to the differential equation above, where c_1, c_2, …, c_k are constants.

One question may arise: What if y_i can be expressed as a linear combination of other solutions? This is resolved by *linear independence*.

Definition 3.1.1: Linear Independence and Dependence

We say that n functions f_1, f_2, …, f_n are **linear dependent** is there are constants c_1, c_2, …, c_n such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0.$$

If the only constants that satisfy the equation

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$, then we say that f_1, f_2, …, f_n are **linear independent**.

For example, $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, and $f_3(x) = 1$ are linear dependent because $f_1(x) + f_2(x) - f_3(x) = \cos^2 x + \sin^2 x - 1 = 0$, but $f_1(x) = x$ and $f_2(x) = |x|$ are linear independent because one function cannot be a constant multiple of another.

Definition 3.1.2: Wronskian

If the functions f_1, f_2, …, f_n have at least $n-1$ derivatives, then the determinant of the matrix

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions f_1, f_2, …, f_n and is denoted by $W(f_1, f_2, \ldots, f_n)$.

Lemma

If there exists $x_0 \in I$ such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then f_1, f_2, …, f_n are linearly independent.

Proof. We prove the contraposition. Suppose f_1, f_2, …, f_n are linearly dependent functions that are at least $n-1$ times differentiable. Then, for some k_1, k_2, …, k_n that are not all zero,

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0.$$

Taking the derivative $n-1$ times, we get

$$\begin{aligned} k_1 f_1 + k_2 f_2 + \cdots + k_n f_n &= 0 \\ k_1 f_1' + k_2 f_2' + \cdots + k_n f_n' &= 0 \\ &\vdots \\ k_1 f_1^{(n-1)} + k_2 f_2^{(n-1)} + \cdots + k_n f_n^{(n-1)} &= 0. \end{aligned}$$

This is a linear system

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since k_1, k_2, …, k_n are not all zero, there exists a nontrivial solution for every x in $(-\infty, \infty)$. Therefore, the determinant of the coefficient matrix, which is the Wronskian, should be zero. ■

The converse is not true: for example, if $f_1(x) = x^2$, and $f_2(x) = x|x|$, then even though $W(f_1, f_2)(x) = 0$, f_1 and f_2 are linear independent. However, if we let f_1, f_2, …, f_n be n solutions to the n-th order linear differential equation, the converse is also true.

Theorem 3.1.3: Existence of Linear Independent Solutions

If y_1, y_2, ..., y_n are n solutions to the n-th order linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0 y = 0,$$

then these solutions are linear independent if and only if

$$W(y_1, y_2, \ldots, y_n)(x) \neq 0$$

for every x.

The proof for if part is done in the lemma above, and the general proof for only if part is omitted. To prove the case when $n = 2$, we use **Abel's identity**.

Lemma : Abel's Identity

If a second-order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

has two solutions y_1 and y_2 on an interval I, then

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}.$$

for each x_0 in the interval I.

Proof. Since $W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$, we have

$$\begin{aligned} W'(y_1, y_2) &= y_1(x)y_2''(x) + y_1'(x)y_2'(x) - y_1'(x)y_2'(x) - y_1''(x)y_2(x) \\ &= y_1(x)y_2''(x) + y_1''(x)y_2(x). \end{aligned}$$

Since y_1 and y_2 are solutions to the equation $y'' + p(x)y' + q(x)y = 0$,

$$\begin{aligned} &y_1(x)y_2''(x) + y_1''(x)y_2(x) \\ &\qquad = y_1(x)\big(-p(x)y_2'(x) - q(x)\big) + y_2(x)\big(-p(x)y_1'(x) - q(x)\big) \\ &\qquad = -p\big(y_1(x)y_2'(x) - y_1'(x)y_2(x)\big) \\ &\qquad = -pW(y_1, y_2)(x). \end{aligned}$$

This is a separable equation about $W(y_1, y_2)(x)$. Solving the equation, we get

$$W(y_1, y_2)(x) = Ce^{-\int_{x_0}^{x} p(t)\,dt}.$$

■

We now prove the $n = 2$ case for only if part of theorem 3.1.3.

Proof. We prove the contraposition. Assume that $W(y_1, y_2)(x_0) = 0$ for some

$x_0 \in I$. Then, $y_1(x)y_2'(x) - y_2(x)y_1'(x) = 0$. Rearranging terms, we get

$$\frac{y_1'(x)}{y_1(x)} = \frac{y_2'(x)}{y_2(x)}$$

as long a $y_1(x)$ and $y_2(x)$ are not zero on every x in I. Without loss of generality, if $y_1(x_0) = 0$ for $x_0 \in I$, then $y_1(x)$ is the unique solution to the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = 0, \quad y'(x_0) = 0.$$

Therefore, $y_1(x)$ and $y_2(x)$ becomes linear dependent. Now, assume that $y_1(x)$ and $y_2(x)$ are never zero on I. Then, integrating both sides gives $\ln|y_1(x)| = \ln|y_2(x)| + c'$, and therefore

$$y_1(x) = Cy_2(x)$$

for any $x \in I$. Therefore, y_1 and y_2 are linear dependent. ■

Definition 3.1.3: Fundamental Set of Solutions

If there are n linear independent solutions y_1, y_2, ..., y_n to the homogeneous linear n-th order differential equation, then the set

$$\{y_1, y_2, \dots, y_n\}$$

is called the **fundamental set of solutions**.

Theorem 3.1.4: General Solution - Homogeneous Equations

Let $y(x)$ be a solution to the homogeneous linear n-th order differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Then, there exists constants c_1, c_2, ..., c_n such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $\{y_1, y_2, \dots, y_n\}$ is the fundamental set of solutions.

We only prove the case for $n = 2$.

Proof. Consider a homogeneous linear 2nd-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

on I, and let $\{y_1, y_2\}$ be the fundamental set of solutions. Let $t \in I$ such that $W(y_1, y_2)(t) \neq 0$, and $g(x)$ be a solution to the equation where $g(t) = a$, and

$g'(t) = b$. Then, since y_1 and y_2 are the basis of $\mathbb{R}^2$, we get

$$c_1 y_1(t) + c_2 y_2(t) = a$$

$$c_1 y_1'(t) + c_2 y_2'(t) = b,$$

which is equal to

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Therefore, there exists unique c_1 and c_2. Define

$$f(x) = c_1 y_1(x) + c_2 y_2(x).$$

Then, $f(x)$ is a solution to the differential equation, and $f(t) = a$, and $f'(t) = b$. Since the solution to the initial-value problem is unique, $y(x) = f(x)$. ■

This general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is called the *complementary solution*, and is denoted by y_c.

Nonhomogeneous Equations

Theorem 3.1.5: Superposition Principle - Nonhomogeneous Equations

Let y_{p_1}, y_{p_2}, …, y_{p_k} be solutions to the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f_1(x),$$
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f_2(x),$$
$$\vdots$$
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f_k(x),$$

respectively. Then,

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is the particular solution to the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f_1(x) + f_2(x) + \cdots + f_k(x).$$

Theorem 3.1.6: General Solution - Nonhomogeneous Equations

Let y_p be any particular solution to the nonhomogeneous linear n-th order differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

Then, the general solution to the differential equation is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x), \end{aligned}$$

where c_1, c_2, …, c_n are constants.

Proof. Let $g(x)$ be any solution to the differential equation above, and y_p be any particular solution.

$$a_n g^{(n)} + a_{n-1} g^{(n-1)} + \cdots + a_1 g' + a_0 g = f(x)$$
$$a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p = f(x)$$

Then, since the equation is linear, by the superposition principle, $g - y_p$ is the solution to the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) - f(x) = 0.$$

However, since the equation above is homogeneous, the solution can be expressed by

$$g(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where c_1, c_2, ..., c_n are constants. Therefore,

$$g(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x). \qquad \blacksquare$$

3.2 Reduction of Order

Reduction of order is a method to find another solution to the homogeneous linear 2nd-order differential equation, with knowing one solution. For instance, suppose there is a homogeneous linear 2nd-order differential equation

$$y'' + f(x)y' + g(x)y = 0.$$

With knowing y_1, the reduction of order method gives a way to find y_2.

Reduction of Order Method

Say that y_1 is a solution to the equation

$$y'' + f(x)y' + g(x)y = 0.$$

We want to find y_2 that is linear independent to y_1. Let $y_2/y_1 = u$, so that $y_2 = uy_1$. Since y_2 should be a solution to the equation above, substituting gives

$$\begin{aligned}\frac{d^2}{dx^2}(uy_1) + f\frac{d}{dx}(uy_1) + g(uy_1) &= (uy_1'' + 2u'y_1' + u''y_1) + f(uy_1' + u'y_1) + g(uy_1)\\ &= u''y_1 + u'(2y_1' + fy_1) + u(y_1'' + fy_1' + gy_1)\\ &= u''y_1 + u'(2y_1' + fy_1)\end{aligned}$$

since $y_1'' + fy_1' + gy_1 = 0$. Let $w = u'$. Then,

$$u''y_1 + u'(2y_1' + fy_1) = w'y_1 + w(2y_1' + fy_1) = 0.$$

Therefore, the equation becomes separable. Solving the equation for w gives

$$\begin{aligned}
y_1 \frac{dw}{dx} &= -(2y_1' + fy_1)w \\
y_1 \frac{1}{w}\, dw &= -(2y_1' + fy_1)\, dx \\
\frac{1}{w}\, dw &= \frac{-(2y_1' + fy_1)}{y_1}\, dx \\
&= -2\frac{y_1'}{y_1} + f\, dx \\
\int \frac{1}{w}\, dw &= \int -2\frac{y_1'}{y_1} - f\, dx \\
&= -2\ln|y_1| - \int f(x)\, dx + c'
\end{aligned}$$

Since

$$\begin{aligned}
\ln|w| + 2\ln|y_1| &== -\int f(x)\, dx + c', \\
wy_1^2 &= c_1 e^{-\int f(x)\, dx}
\end{aligned}$$

and

$$w = c_1 \frac{e^{-\int f(x)\, dx}}{y_1^2}$$

where $c_1 = \pm e^{c'}$. Since, $w = u'$, integrating both sides gives

$$u = c_1 \int \frac{e^{-\int f(x)\, dx}}{y_1^2} + c_2\, dx.$$

Since c_1 and c_2 are constants, we choose $c_1 = 1$ and $c_2 = 0$ so that u does not become a constant because if u is a constant, then $y_1(x)$ and $y_2(x)$ are linearly dependent. Finally, we get

$$\begin{aligned}
y_2 &= uy_1 \\
&= y_1 \int \frac{e^{-\int f(x)\, dx}}{y_1^2}\, dx.
\end{aligned}$$

Example 1

Find the general solution of $y'' + 4y = 0$, knowing that $\cos 2x$ is a solution.

Solution In this case, $f(x) = 0$. Substituting $y_1 = \cos 2x$ gives

$$\begin{aligned} y_2 &= y_1 \cdot \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx \\ &= \cos 2x \int \frac{e^c}{\cos^2 2x}\,dx \\ &= e^c \cos 2x \int \sec^2 2x\,dx \\ &= \frac{e^c}{2} \cos 2x \tan 2x \\ &= \frac{e^c}{2} \sin 2x. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= c_1 \cos 2x + c_2' \cdot \frac{e^c}{2} \sin 2x \\ &= c_1 \cos 2x + c_2 \sin 2x \end{aligned}$$

where $c_2 = c_2' \cdot e^c/2$.

Example 2

Find the general solution of $y'' + 3y' - 4y = 0$, knowing that e^x is a solution.

Solution In this case, $f(x) = 3$. Substituting $y_1 = e^x$ gives

$$\begin{aligned} y_2 &= y_1 \cdot \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx \\ &= e^x \int \frac{e^{-3x}}{e^{2x}}\,dx \\ &= e^x \int e^{-5x}\,dx \\ &= e^x \cdot \left(-\frac{1}{5} e^{-5x} \right) \\ &= -\frac{1}{5} e^{-4x}. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= c_1 e^x + c_2' \cdot \left(-\frac{1}{5} e^{-4x} \right) \\ &= c_1 e^x + c_2 e^{-4x}. \end{aligned}$$

3.3 Homogeneous Linear Equations with Constant Coefficients

This section covers homogeneous linear equations with constant coefficients. That is, equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where a_n, a_{n-1}, ..., a_0 are constants. We first consider the 2nd-order case, where the equation is

$$ay'' + by' + cy = 0.$$

2nd-Order Equations

Since the equation is linear and 2nd-order, there are two linear independent solutions. We claim that the solutions are of the form $e^{\alpha x}$, in general. Substituting $e^{\alpha x}$ gives

$$(a\alpha^2 + b\alpha + c)e^{\alpha x} = 0,$$

which gives $a\alpha^2 + b\alpha + c = 0$ since exponential functions are always positive. This typical quadratic equation is called the **characteristic equation**.

Case 1: Distinct Real Roots

When the quadratic equation has two distinct real roots α and β, then the solution to the differential equation is $e^{\alpha x}$ and $e^{\beta x}$. These two solutions are linearly independent because one cannot be a constant multiple of another. Therefore, the general solution is

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x}.$$

Case 2: Repeated Real Roots

When the quadratic equation has repeated real roots α, then we know one solution, but we need one more. To get a second solution, we use the reduction of order method. Recall the reduction of order formula

$$y_2 = y_1 \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx,$$

where y_1 is a known solution. Here, $f(x) = b/a = -2\alpha$ by Vieta's formula.

Substituting $e^{\alpha x}$ to the formula gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx \\ &= e^{\alpha x} \int \frac{e^{-\int -2\alpha\,dx}}{e^{2\alpha x}}\,dx \\ &= e^{\alpha x} \int dx \\ &= xe^{\alpha x}, \end{aligned}$$

which is the second solution that is linear independent of the first. Therefore, the general solution is

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x}.$$

Case 3: Complex Conjugate Roots
If the quadratic equation has complex roots $a+bi$, then its conjugate $a-bi$ is also a root. Therefore, two solutions are $e^{(a+bi)x}$ and $e^{(a-bi)x}$, which is a problem since there are complex numbers in exponents. For this, we use the **Euler's Formula** which states that

$$e^{ix} = \cos x + i \sin x.$$

It follows that

$$e^{ibx} = \cos bx + i \sin bx \text{ and } e^{-ibx} = \cos bx - i \sin bx.$$

Therefore,

$$\begin{aligned} y_1 &= e^{(a+bi)x} \\ &= e^{ax} \cdot e^{bix} \\ &= e^{ax}(\cos bx + i \sin bx), \end{aligned}$$

and

$$y_2 = e^{ax}(\cos bx - i \sin bx).$$

Since the equation is homogeneous, any linear combination of y_1 and y_2 is also a solution. Therefore,

$$\begin{aligned} Y_1 &= \frac{1}{2}(y_1 + y_2) = e^{ax} \cos bx \text{ and} \\ Y_2 &= \frac{1}{2i}(y_1 - y_2) = e^{ax} \sin bx \end{aligned}$$

are two solutions to the equation. By theorem 3.1.3, these two solutions are linearly independent because

$$\begin{aligned} W(e^{ax}\cos bx, e^{ax}\sin bx)(x) \\ &= \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ ae^{ax}\cos bx - be^{ax}\sin bx & ae^{ax}\sin bx + be^{ax}\cos bx \end{vmatrix} \\ &= be^{2ax} \neq 0. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= c_1Y_1 + c_2Y_2 \\ &= c_1e^{ax}\cos bx + c_2e^{ax}\sin x \\ &= e^{ax}(c_1\cos bx + c_2\sin bx). \end{aligned}$$

Example 1

Find the general solution to the equation $y'' - k^2y = 0$, where k is a constant.

Solution The characteristic equation for $y'' - k^2y = 0$ is $\alpha^2 - k^2 = 0$, and therefore $\alpha = \pm k$. Thus, the general solution is

$$y = c_1e^{kx} + c_2e^{-kx}.$$

Example 2

Find the general solution to the equation $y'' - 2ky' + k^2y = 0$, where k is a constant.

Solution The characteristic equation for $y'' - 2ky' + k^2y = 0$ is $\alpha^2 - 2k\alpha + k^2 = 0$, and therefore $\alpha = k$ of multiplicity 2. Thus, the general solution is

$$y = c_1e^{kx} + c_2xe^{kx}.$$

Example 3

Find the general solution to the equation $y'' + k^2y = 0$, where k is a constant.

Solution The characteristic equation for $y'' + k^2y = 0$ is $\alpha^2 + k^2 = 0$, and therefore $\alpha = \pm ik$. Thus, the general solution is

$$y = c_1\cos kx + c_2\sin kx.$$

Higher Order Equations

For higher order equations of the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

the characteristic equation is

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0.$$

By **Fundamental Theorem of Algebra**, which states that an equation with degree n has n solutions in $\mathbb{C}$, one can find n values of α, and thus n linear independent solutions. If a root α_1 has multiplicity k, then the k linear independent solutions are

$$y = e^{\alpha_1 x},\ xe^{\alpha_1 x},\ x^2e^{\alpha_1 x},\ \ldots,\ x^{k-1}e^{\alpha_1 x}.$$

Theorem 3.3.1: Characteristic Equation with Multiplicity k

If a characteristic equation of a homogeneous linear differential equation with constant coefficients has a root α_1 with multiplicity k, then the k linear independent solutions to the differential equation are

$$y = e^{\alpha_1 x},\ xe^{\alpha_1 x},\ x^2e^{\alpha_1 x},\ \ldots,\ x^{k-1}e^{\alpha_1 x}.$$

Proof. Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$. Then,

$$L(e^{\alpha x}) = (\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0)e^{\alpha x}.$$

Let α_1 be a root with multiplicity k. Then, the characteristic equation can be expressed as $(\alpha-\alpha_1)^k f(\alpha)$, where $f(\alpha) = (\alpha-\alpha_2)\cdots(\alpha-\alpha_{n-k})$. So we get

$$L(e^{\alpha x}) = (\alpha-\alpha_1)^k f(\alpha)e^{\alpha x}.$$

The differentiations with respect to α and x are independent, so differentiating both sides with respect to α gives

$$\begin{aligned}\frac{\partial}{\partial\alpha}L(e^{\alpha x}) &= L\left(\frac{\partial}{\partial\alpha}e^{\alpha x}\right)\\ &= L(xe^{\alpha x})\\ &= k(\alpha-\alpha_1)^{k-1}f(\alpha)e^{\alpha x} + (\alpha-\alpha_1)^k\frac{\partial}{\partial\alpha}\big(f(\alpha)e^{\alpha x}\big)\\ &= 0\end{aligned}$$

for $\alpha = \alpha_1$. Therefore, $xe^{\alpha x}$ is also a solution to the differential equation. Repeating this procedure $k-1$ times, the right side is always zero since it contains a $\alpha-\alpha_1$ term. The degree of x on the left side increases every time, taking the derivative of the left side. Therefore, $x^ie^{\alpha x}$ is a solution for i =1, 2, …, $k-1$. ∎

Example 4

Solve $y^{(5)} + y^{(4)} - y' - y = 0$.

Solution The characteristic equation is

$$\alpha^5 + \alpha^4 - \alpha - 1 = 0.$$

Solving for α gives

$$(\alpha - 1)(\alpha + 1)^2(\alpha^2 + 1) = 0$$
$$\alpha = 1, -1, \pm i,$$

where -1 has multiplicity 2. Therefore, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x} + c_4 \sin x + c_5 \cos x.$$

3.4 Cauchy-Euler Equations

Definition 3.4.1: Cauchy-Euler Equation

Linear differential equations of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 x y = 0,$$

where a_n, a_{n-1}, …, a_0 are constants, are called **Cauchy-Euler equations**.

For each term, the order of y and the degree of x should be the same. Also for this section, we first consider the 2nd-order case, where the equation is

$$ax^2 y'' + bxy' + cy = 0.$$

2nd-Order Equations

Since the equation is linear and 2nd-order, there are two linearly independent solutions. We claim that the solutions are of the form x^α, in general. Substituting x^α gives

$$\big(a\alpha(\alpha - 1) + b\alpha + c\big)x^\alpha = 0,$$

which gives $a\alpha(\alpha - 1) + b\alpha + c = 0$ or $a\alpha^2 + (-a + b)\alpha + c = 0$. This is the characteristic equation.

Case 1: Distinct Real Roots

When the quadratic equation has two distinct real roots α and β, then the solution to the differential equation is x^α and x^β. These two solutions are linearly independent because one cannot be a constant multiple of another. Therefore, the general solution is

$$y = c_1 x^\alpha + c_2 x^\beta.$$

Case 2: Repeated Real Roots
When the quadratic equation has repeated real roots α, then we know one solution, but we need one more. To get a second solution, we use the reduction of order method. Recall the reduction of order formula

$$y_2 = y_1 \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx,$$

where y_1 is a known solution. Here, $f(x) = bx/ax^2 = b/ax$, and $(b-a)/a = -2\alpha$ by Vieta's formula. Substituting x^α to the formula gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int f(x)\,dx}}{y_1^2}\,dx \\
&= x^\alpha \int \frac{e^{-\int (b/ax)\,dx}}{x^{2\alpha}} \\
&= x^\alpha \int \frac{e^{-b\ln x/a}}{x^{-(b-a)/a}}\,dx \\
&= x^\alpha \int \frac{1}{x}\,dx \\
&= x^\alpha \ln x.
\end{aligned}$$

which is the second solution that is linear independent of the first. Therefore, the general solution is

$$y = c_1 x^\alpha + c_2 x^\alpha \ln x.$$

Case 3: Complex Conjugate Roots
If the quadratic equation has complex roots $a+ib$, then its conjugate $a-ib$ is also a root. Therefore, two solutions are x^{a+ib} and x^{a-ib}. To solve the problem for complex exponential, we change the expression to

$$\begin{aligned}
x^{a+ib} &= x^a \cdot x^{ib} \\
&= x^a \cdot (e^{\ln x})^{ib} \\
&= x^a \cdot e^{ib\ln x} \\
&= x^a(\cos b\ln x + i\sin b\ln x) \text{ and} \\
x^{a-ib} &= x^a(\cos b\ln x - i\sin b\ln x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_1 &= x^a(\cos b\ln x + i\sin b\ln x) \text{ and} \\
y_2 &= x^a(\cos b\ln x - i\sin b\ln x).
\end{aligned}$$

Since the equation is homogeneous, any linear combination of y_1 and y_2 is also a solution. Therefore,

$$Y_1 = \frac{1}{2}(y_1 + y_2) = x^a \cos b \ln x \text{ and}$$
$$Y_2 = \frac{1}{2i}(y_1 - y_2) = x^a \sin b \ln x$$

are two solutions to the equation. By theorem 3.1.3, these two solutions are linearly independent because

$$\begin{aligned} &W(x^a \cos b \ln x, x^a \sin b \ln x)(x) \\ &= \begin{vmatrix} x^a \cos b \ln x & x^a \sin b \ln x \\ -bx^a \sin b \ln x + ax^{a-1} \cos b \ln x & bx^a \cos b \ln x + ax^{a-1} \sin b \ln x \end{vmatrix} \\ &= bx^{2a}(\sin^2 b \ln x + \cos^2 b \ln x) \\ &= bx^{2a} \neq 0. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y &= c_1 Y_1 + c_2 Y_2 \\ &= c_1 x^a \cos b \ln x + c_2 x^a \sin b \ln x \\ &= x^a (c_1 \cos b \ln x + c_2 \sin b \ln x). \end{aligned}$$

Example 1

Find the general solution to the equation $x^2 y'' + xy' - 4y = 0$.

Solution The characteristic equation for $x^2y'' + xy' - 4y = 0$ is $\alpha(\alpha - 1) + \alpha - 4 = \alpha^2 - 4 = 0$, and therefore $\alpha = \pm 2$. Thus, the general solution is

$$y = c_1 x^2 + c_2 x^{-2}.$$

Example 2

Find the general solution to the equation $x^2 y'' - xy' + y = 0$.

Solution The characteristic equation for $x^2y'' - xy' + y = 0$ is $\alpha(\alpha - 1) - \alpha + 1 = \alpha^2 - 2\alpha + 1 = 0$, and therefore $\alpha = 1$ of multiplicity 2. Thus, the general solution is

$$y = c_1 x + c_2 x \ln x.$$

Example 3

Find the general solution to the equation $x^2 y'' + xy' + 4y = 0$.

Solution The characteristic equation for $x^2y'' + xy' - 4y = 0$ is $\alpha(\alpha-1) + \alpha + 4 = \alpha^2 + 4 = 0$, and therefore $\alpha = \pm 2i$. Thus, the general solution is

$$y = c_1 \cos 2\ln x + c_2 \sin 2\ln x.$$

Higher Order Equations

For higher order equations of the form

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = 0,$$

the characteristic equation is

$$a_n\alpha(\alpha-1)\cdots(\alpha-n+1) + \cdots + a_1\alpha + a_0 = 0.$$

By Fundamental Theorem of Algebra, one can get n values of α, and thus n linear independent solutions. If a root α_1 has multiplicity k, then the k linear independent solutions are

$$y = x^{\alpha_1},\ x^{\alpha_1}\ln x,\ x^{\alpha_1}(\ln x)^2,\ \ldots,\ x^{\alpha_1}(\ln x)^{k-1}.$$

Theorem 3.4.1: Characteristic Equation with Multiplicity k

If a characteristic equation of a Cauchy-Euler equation has a root α_1 with multiplicity k, then the k linear independent solutions to the differential equation are

$$y = x^{\alpha_1},\ x^{\alpha_1}\ln x,\ x^{\alpha_1}(\ln x)^2,\ \ldots,\ x^{\alpha_1}(\ln x)^{k-1}.$$

The proof is omitted, but one can prove by substituting $x = e^t$ which changes the Cauchy-Euler equation to a homogeneous linear equation with constant coefficients.

Example 4

Solve $x^3y''' - xy' - 3y = 0$.

Solution The characteristic equation is

$$\begin{aligned}\alpha(\alpha-1)(\alpha-2) - \alpha - 3 &= \alpha^3 - 3\alpha^2 + \alpha - 3\\ &= (\alpha-3)(\alpha^2+1) = 0.\end{aligned}$$

Solving for α gives $\alpha = 3, \pm i$. Therefore, the general solution is

$$y = c_1x^3 + c_2\cos\ln x + c_3\sin\ln x.$$

3.5 Undetermined Coefficients

Recall from section 3.1 that to solve a differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

one should find the complementary and particular solutions to the equation, and add them. For the last few sections, we have covered how to find complementary solutions for some equations. This section and the next section cover how to find particular solutions. Although the variation of parameter method is used more, the undetermined coefficient method is also worth knowing.

Undetermined Coefficient Method

Suppose there is an equation of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x),$$

where a_n, a_{n-1}, $\ldots$, a_0 are constants. If $f(x)$ is either a polynomial function, exponential function, trigonometric function, or a finite sum or product of these, we can *guess* the particular solution. This is best explained by an example.

Example 1

Find a particular solution of $y'' + 3y' + 2y = 2x^2 + 6x + 4$.

Solution Since the right-hand side is a polynomial of degree 2, we can guess that y is a polynomial of degree 2. Therefore, let $y_p = ax^2 + bx + c$. Then,

$$\begin{aligned} y'' + 3y' + 2y &= (ax^2 + bx + c)'' + 3(ax^2 + bx + c)' + 2(ax^2 + bx + c) \\ &= 2a + 3(2ax + b) + 2(ax^2 + bx + c) \\ &= 2ax^2 + (6a + 2b)x + 2a + 3b + 2c = 2x^2 + 6x + 4. \end{aligned}$$

This gives a system of linear equations

$$\begin{aligned} 2a \qquad\quad &= 2 \\ 6a + 2b \quad &= 6 \\ 2a + 3b + 2c &= 4. \end{aligned}$$

Solving the linear system, we get $a = 1$, $b = 0$, and $c = 1$. Therefore, $y_p = x^2 + 1$. There may be other particular solutions. However, we only need one particular solution and can say that the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + x^2 + 1.$$

The table below shows which form of y_p to try corresponding to $f(x)$.

Guessing Particular Solutions	
$f(x)$	y_p
Polynomial of degree n (assuming the equation contains y)	Polynomial of the same degree
e^{kx}	ae^{kx}
$\sin kx$	$a \sin kx + b \sin kx$
$\cos kx$	$a \sin kx + b \sin kx$

If $f(x)$ is a finite sum or product of these functions, one can try the particular solution as the sum or product of each corresponding y_p.

Example 2

Solve $y'' - 2y' + y = 25 \sin 2x + (x + 6)e^{3x}$.

Solution For the complementary solution, since the characteristic equation is $\alpha^2 - 2\alpha + 1 = 0$, $\alpha = 1$ with multiplicity 2. The solution is

$$y_c = c_1 e^x + c_2 x e^x.$$

For the particular solution, let the particular solution to the equation

$$y'' - 2y' + y = 25 \sin 2x$$

be y_{p_1} and the particular solution to the equation

$$y'' - 2y' + y = (x + 6)e^{3x}$$

be y_{p_2}. Then, $y_p = y_{p_1} + y_{p_2}$ by superposition principle. We first guess $y_1 = a_1 \sin 2x + b_1 \cos 2x$. Substituting to the equation gives

$$\begin{aligned}
&y'' - 2y' + y \\
&\quad = (a_1 \sin 2x + b_1 \cos 2x)'' - 2(a_1 \sin 2x + b_1 \cos 2x)' + (a_1 \sin 2x + b_1 \cos 2x) \\
&\quad = (-4a_1 \sin 2x - 4b_1 \cos 2x) - 2(2a_1 \cos 2x - 2b_1 \sin 2x) + (a_1 \sin 2x + b_1 \cos 2x) \\
&\quad = (-3a_1 + 4b_1) \sin 2x + (-4a_1 - 3b_1) \cos 2x = 25 \sin 2x,
\end{aligned}$$

and a system of linear equations

$$\begin{aligned}
-3a_1 + 4b_1 &= 25 \\
-4a_1 - 3b_1 &= 0.
\end{aligned}$$

Therefore, $a_1 = -3$, $b_1 = 4$ and $y_{p_1} = -3\sin 2x + 4\cos 2x$. Now, for y_{p_2}, since $(x+6)e^{3x}$ is a product of a linear function and an exponential function, we guess $y_{p_2} = (a_2 x + b_2)e^{3x}$. Substituting to the equation gives

$$\begin{aligned}
&y'' - 2y' + y \\
&\quad = \big((a_2x+b_2)e^{3x}\big)'' - 2\big((a_2x+b_2)e^{3x}\big)' + \big((a_2x+b_2)e^{3x}\big) \\
&\quad = \big(9(a_2x+b_2)e^{3x} + 6a_2e^{3x}\big) - 2\big(3(a_2x+b_2)e^{3x} + a_2e^{3x}\big) + \big((a_2x+b_2)e^{3x}\big) \\
&\quad = \big(4a_2x + (4a_2+4b_2)\big)e^{3x},
\end{aligned}$$

and a system of linear equations

$$\begin{aligned}
4a_2 \qquad &= 1 \\
4a_2 + 4b_2 &= 6.
\end{aligned}$$

Solving the system, we get $a_2 = 1/4$, $b_2 = 5/4$ and $y_{p_2} = (x/4+5/4)e^{3x}$. Therefore, the general solution is

$$y = c_1e^x + c_2xe^x - 3\sin 2x + 4\cos 2x + \frac{xe^{3x}}{4} + \frac{5e^{3x}}{4}.$$

For most of the cases, particular solutions can be guessed by following the rule in the table above, but there are some exceptions. We give an example.

Example 3

Show that, for any real a, ae^{2x} cannot be a particular solution to the equation $y'' - 5y' + 6y = 3e^{2x}$.

Solution If we substitute $y = ae^{2x}$ to the equation, we get

$$\begin{aligned}
y'' - 5y' + 6y &= (ae^{2x})'' - 5(ae^{2x})' + 6(ae^{2x}) \\
&= 4ae^{2x} - 10ae^{2x} + 6ae^{2x} \\
&= 0 \neq 3e^{2x}.
\end{aligned}$$

This happens because e^{2x} is already included in the complementary solution. The complementary solution for the equation above is

$$y = c_1e^{2x} + c_2e^{3x},$$

and $ae^{2x} = a\cdot e^{2x} + 0\cdot e^{3x}$, so it contradicts the definition of a particular solution. To resolve this, we try the form $y_p = ax^ne^{2x}$. The smallest n to make y_p not a complementary solution is $n = 1$, so we guess $y_p = axe^{2x}$.

Example 4
Solve $y'' - 5y' + 6y = 3e^{2x}$.

Solution Substituting $y = axe^{2x}$ to the equation, we get

$$\begin{aligned} y'' - 5y' + 6y &= (axe^{2x})'' - 5(axe^{2x}) + 6(axe^{2x}) \\ &= (4axe^{2x} + 4ae^{2x}) - 5(ae^{2x} + 2axe^{2x}) + 6(axe^{2x}) \\ &= 5ae^{2x} = 3e^{2x} \end{aligned}$$

Therefore, $5a = 3$, and $a = 3/5$. The general solution to the equation is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{3}{5}xe^{2x}.$$

To guess the particular solution, if $p(x)$ is one of the trial particular solution formula to the corresponding $f(x)$ in the table above, then one should use $y_p = x^n p(x)$, where n is the least positive integer such eliminates the duplication with the complementary solution.

Example 5
Solve $y'' + 10y' + 25y = (3x + 4)e^{-5x}$.

Solution For the complementary solution, since the characteristic equation is $\alpha^2 + 10\alpha + 25 = 0$, $\alpha = -5$ with multiplicity 2. The solution is

$$y_c = c_1e^{-5x} + c_2xe^{-5x}.$$

Since $f(x) = (3x + 4)e^{-5x}$, the form of the particular solution should be $(ax + b)e^{-5x}$. However, since this can be expressed by a linear combination of e^{-5x} and xe^{-5x}, we multiply x to make y_p linear independent with y_c. Therefore, we use $y_p = (ax^3 + bx^2)e^{-5x}$. Substituting to the equation gives

$$\begin{aligned} &y'' + 10y' + 25y \\ &\quad= \big((ax^3 + bx^2)e^{-5x}\big)'' + 10\big((ax^3 + bx^2)e^{-5x}\big)' + \big((ax^3 + bx^2)e^{-5x}\big) \\ &\quad= (25ax^3e^{-5x} - 30ax^2e^{-5x} + 25bx^2e^{-5x} + 6axe^{-5x} - 20bxe^{-5x} + 2be^{-5x}) \\ &\quad+ 10(-5ax^3e^{-5x} + 3ax^2e^{-5x} - 5bx^2e^{-5x} + 2bxe^{-5x}) + 25(ax^3e^{-5x} + bx^2e^{-5x}) \\ &\quad= (6ax + 2b)e^{-5x}, \end{aligned}$$

and therefore $6a = 3$, $2b = 4$. The solution to the equation is

$$y = c_1e^{-5x} + c_2xe^{-5x} + \frac{x^3e^{-5x}}{2} + 2x^2e^{-5x}.$$

3.6 Variation of Parameters

The variation of parameter method, already introduced in solving first-order linear equations, is another method for finding the particular solution. The variation of parameter method is better than the undetermined coefficient method in general because it always yields a particular solution. Before we start, we introduce **Cramer's rule**, which will help to find the particular solution.

Theorem 3.6.1: Cramer's Rule

Consider a linear system $A\mathbf{x} = \mathbf{b}$ with n equations and n variables, where $\det A \neq 0$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Then,

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is the matrix obtained by replacing ith column by $\mathbf{b}$.

Proof. Let C be the cofactor matrix of A. Then, by definition, C^T is the adjugate matrix of A. Therefore,

$$A \cdot C^T = \det A \cdot I_n.$$

Since $\det A \neq 0$, A is invertible, and

$$A^{-1} = \frac{1}{\det A} C^T.$$

The solution to the linear system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{aligned} \mathbf{x} &= A^{-1}\mathbf{b} \\ &= \left(\frac{1}{\det A} C^T \right) \mathbf{b}. \end{aligned}$$

By the definition of matrix product, we have

$$x_i = \frac{1}{\det A} \left(\sum_{j=1}^{k} A_{ji} b_j \right).$$

Since A_i only differs with ith column with A, the matrix obtained by deleting ith column and jth row from A_i is equal to the matrix obtained by deleting ith column and jth row from A. Therefore, $A_{ji} = [A_i]_{ji}$. We finally have

$$\begin{aligned} x_i &= \frac{1}{\det A}\left(\sum_{j=1}^{k} A_{ji} b_j\right) \\ &= \frac{1}{\det A}\left(\sum_{j=1}^{k} [A_i]_{ji} b_j\right) \\ &= \frac{1}{\det A} \cdot \det A_i \\ &= \frac{\det A_i}{\det A}. \end{aligned}$$

■

2nd-Order Case

We first consider a 2nd-order case. Consider a 2nd-order linear differential equation

$$y'' + p(x)y' + q(x) = f(x).$$

Let the fundamental set of solutions be $\{y_1, y_2\}$. We set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, and look for $u_1(x)$ and $u_2(x)$. Substituting to the equation gives

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (u_1y_1 + u_2y_2)'' + p(u_1y_1 + u_2y_2)' + q(u_1y_1 + u_2y_2) \\ &= (u_1y_1'' + 2u_1'y_1' + u_1''y_1 + u_2y_2'' + 2u_2'y_2' + u_2''y_2) \\ &+ p(u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2) + q(u_1y_1 + u_2y_2) \\ &= u_1(y_1'' + py_1' + qy_1) + 2u_1'y_1' + u_1''y_1 \\ &+ u_2(y_2'' + py_2' + qy_2) + 2u_2'y_2' + u_2''y_2 + pu_1'y_1 + pu_2'y_2 \\ &= u_1'y_1' + u_2'y_2' + u_1'y_1' + u_2'y_2' + u_1''y_1 + u_2''y_2 + pu_1'y_1 + pu_2'y_2 \\ &= u_1'y_1' + u_2'y_2' + \frac{d}{dx}(u_1'y_1 + u_2'y_2) + p(u_1'y_1 + u_2'y_2) = f(x). \end{aligned}$$

Since we need only one pair of $\big(u_1(x), u_2(x)\big)$, we assume, for simplicity, that $u_1'y_1 + u_2'y_2 = 0$. Then, we get

$$u_1'y_1' + u_2'y_2' + \frac{d}{dx}(u_1'y_1 + u_2'y_2) + p(u_1'y_1 + u_2'y_2) = u_1'y_1' + u_2'y_2' = f(x)$$

We now have two equations

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x), \end{aligned}$$

where y_1 and y_2 are known. Notice that this system can be expressed as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.$$

Since y_1 and y_2 are linear independent, $\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = W(y_1, y_2)(x) \neq 0$, and therefore the system has a unique solution. Using Cramer's rule, we get

$$u_1' = \frac{\det \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = -\frac{y_2 f(x)}{W(y_1, y_2)(x)} \text{ and}$$

$$u_2' = \frac{\det \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{y_1 f(x)}{W(y_1, y_2)(x)}.$$

Integrating each formula gives u_1 and u_2, and one can find the particular solution to the equation.

Example 1

Solve $y'' + y = \tan x$.

Solution To find the complementary solution, we solve the characteristic equation $\alpha^2 + 1 = 0$, which gives $\alpha = \pm i$. Therefore, $y_1 = \cos x$, $y_2 = \sin x$, and the complementary solution is

$$y = c_1 \cos x + c_2 \sin x.$$

To find the particular solution, we let $y_p = u_1 \cos x + u_2 \sin x$. Then,

$$\begin{aligned} u_1' &= -\frac{y_2 f(x)}{W(y_1, y_2)(x)} \\ &= -\frac{\sin x \tan x}{1} \\ &= -\sin x \tan x \\ &= -\frac{\sin^2 x}{\cos x} \\ &= -\frac{1 - \cos^2 x}{\cos x} \\ &= -\sec x + \cos x \end{aligned}$$

and

$$\begin{aligned} u_2' &= \frac{y_1 f(x)}{W(y_1, y_2)(x)} \\ &= \frac{\cos x \tan x}{1} \\ &= \sin x. \end{aligned}$$

Integrating both expressions gives

$$u_1 = -\ln(\tan x + \sec x) + \sin x \text{ and } u_2 = -\cos x.$$

Therefore, the particular solution is

$$\begin{aligned} y_p &= \big(-\ln(\tan x + \sec x) + \sin x\big)\cos x - \cos x \sin x \\ &= -\cos x \ln(\tan x + \sec x), \end{aligned}$$

and the general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln(\tan x + \sec x).$$

Higher Order Equations

The variation of parameter method can also be used in equations with higher order. Consider a linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

where the fundamental set of solutions is $\{y_1, y_2, \ldots, y_n\}$. We let $y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n$ and solve the linear system

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \ddots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddots \\ f(x) \end{bmatrix}$$

to get u_1', u_2', …, u_n'. Integrating each of these will yield a particular solution.

3.7 Nonlinear Equations

Nonlinear equations, in general, are harder than linear equations. First, they don't satisfy the superposition principle. For instance, consider a nonlinear equation $(y')^2 - y^2 = 0$. Solving this gives

$$
\begin{aligned}
(y')^2 &= y^2 \\
y' &= \pm y \\
y &= e^x, e^{-x}.
\end{aligned}
$$

Therefore, $y = e^x$ and $y = e^{-x}$ are two solutions to the equation. However, $y = e^x + e^{-x}$ is not a solution to the equation above because

$$
\begin{aligned}
(y')^2 - y^2 &= (e^x - e^{-x})^2 - (e^x + e^{-x})^2 \\
&= -4 \neq 0.
\end{aligned}
$$

Plus, the solution to nonlinear equations may not even exist, and may not be unique if it exists. Still, some nonlinear equations can be solved by substituting appropriate formulas.

Reduction of Order

Nonlinear equations of second-order can be reduced to first-order equations under some conditions. Two cases where the reduction of order method will work are when x is not included in the equation or when y is not included in the equation.

When y is not included in the equation, i.e. when the equation is of the form $f(x, y', y'') = 0$, then substituting $u = y'$ will reduce the equation to first-order. Since $y'' = u'$, the equation becomes $f(x, u, u') = 0$, which is first-order.

Example 1

Solve $y''y' = -\dfrac{1}{x^3}$, $y(1) = 1$, $y'(1) =$

Solution Let $u = y'$. Then, $y'' = u'$, and the equation becomes

$$u'u = -\frac{1}{x^3},$$

which is separable. Solving for u gives

$$\begin{aligned} u\frac{du}{dx} &= -\frac{1}{x^3} \\ u\,du &= -\frac{1}{x^3}\,dx \\ \int u\,du &= -\int \frac{1}{x^3}\,dx \\ \frac{1}{2}u^2 &= \frac{1}{2}x^{-2} + c_1. \end{aligned}$$

Since $y'(1) = u(1) = 1$, $c_1 = 0$, and therefore $u = y' = x^{-1}$. Finally, solving for y gives

$$y = \ln x + c_2.$$

The initial condition $y(1) = 1$ gives $c_2 = 1$, and hence the solution is

$$y = \ln x + 1.$$

When x is not included in the equation, i.e. when the equation is of the form $f(y, y', y'') = 0$, then substituting $u = y'$ will reduce the equation to first-order with respect to y. Since $y'' = u'$,

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = y'\frac{du}{dy}.$$

Therefore, the equation becomes $f(y, u, u\,du/dy)$, which is first-order with respect to y.

Example 2

Solve $yy'' + (y')^2 = 0$.

Solution Let $u = y'$. Then, $y'' = u\,du/dy$. Substituting these to the original equation gives

$$\begin{aligned} yu\frac{du}{dy} + u^2 &= 0 \\ -y\frac{du}{dy} &= u \\ \int \frac{1}{u}du &= -\int \frac{1}{y}dy \\ \ln|u| &= -\ln|y| + c_1 \end{aligned}$$

and therefore
$$u = c_2 \frac{1}{y}$$
where $c_2 = \pm e^{c_1}$. Since $u = y'$, solving for y gives
$$\frac{dy}{dx} = c_2 \frac{1}{y}$$
$$y\,dy = c_2\,dx$$
$$\int y\,dy = c_2 \int dx$$
$$\frac{1}{2} y^2 = c_2 x + c_3.$$
Therefore, the solution to the equation is
$$y = \sqrt{c_2' x + c_3'}$$
where $c_2' = 2c_2$ and $c_3' = 2c_3$.

Besides the reduction of order method, some nonlinear equations can be solved by applying an appropriate substitution. There isn't a specific answer to which substitution one should make, and one should find out which substitution helps to make the equation simpler.

Example 3

Solve $\dfrac{dy}{dx} = x^3(x+y)^2 - \dfrac{3(x+y)}{x} - 1.$

Solution Let $u = x + y$. Then, $\dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$. Then, the equation becomes
$$\frac{du}{dx} = x^3 u^2 - \frac{3u}{x}$$
and thus,
$$\frac{du}{dx} + \frac{3}{x} u = x^3 u^2$$
which is Bernoulli's equation. Now let $v = u^{-1}$. Then, since $dv = -u^{-2} du$, we get
$$u^{-2} \frac{du}{dx} + \frac{3}{x} u^{-1} = x^3$$
$$-\frac{dv}{dx} + \frac{3}{x} v = x^3$$
$$\frac{dv}{dx} - \frac{3}{x} v = -x^3.$$

The integrating factor is

$$\mu(x) = e^{\int -\frac{3}{x}\,dx} = x^{-3},$$

and hence

$$\begin{aligned} v &= x^3 \int x^3 \cdot (-x^{-3})\,dx \\ &= x^3 \int -1\,dx \\ &= -x^4 + cx^3. \end{aligned}$$

Since $v = u^{-1} = \dfrac{1}{x+y}$,

$$x + y = \frac{1}{-x^4 + cx^3}$$

and therefore the solution is

$$y = -x + \frac{1}{-x^4 + cx^3}.$$

Chapter 4

Series Solutions

Until now, we have covered linear differential equations with constant coefficients, and Cauchy-Euler equations. There are other equations, where they have variable coefficients, such as $y''' - x^2y' = 0$. Some of these equations can not be solved explicitly, so we use an analytic method, called the *power series method.* The idea is to find a solution in terms of power series. Handling power series is much easier than exponential or trigonometric functions because they are polynomials. This chapter covers the power series method to solve differential equations.

4.1 The Power Series

Definition 4.1.1: Power Series

A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots.$$

where a_i is the coefficient of the ith term, and c is a constant.

The definition above specifically illustrates a power series centered at c. If $c = 0$, then the power series can be expressed by

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Definition 4.1.2: Convergence

A sequence $\{s_n\}$ **converges** to L if for every $\epsilon > 0$ there exists N such that $|s_n - L| < \epsilon$ for all $n > N$.

If we define $\{s_n\}$ where $s_i = \sum_{n=0}^{i} a_n(x-c)^n$. Then, the power series converges if and only if s_n converges. The interval of x which makes the series converge is called the *interval of convergence.*

Definition 4.1.3: Interval of Convergence

The **interval of convergence** of a power series is the interval where for any x in the interval, the series converges for x. The interval is usually denoted by $|x - c| < R$, where R is called the **radius of convergence**.

This states if x is apart from c by a distance smaller than R, then the series converges. To find R, usually the **ratio test** is used.

Theorem 4.1.1: Ratio Test

Consider a sequence $\{s_n\}$. If we define

$$L = \lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right|,$$

then the series $\sum_{n=0}^{\infty} s_n$

- absolutely converges if $L < 1$,
- diverges if $L > 1$,
- is inconclusive if $L = 1$.

To determine if a power series converges or not, we define $\{s_n\}$ where $s_n = a_n(x-c)^n$. Using the ratio test, we get

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| &= \lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)}{a_n} \right| \\ &= |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.\end{aligned}$$

Therefore, since L should be smaller than 1 to make the series converge, we have

$$|x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

which can determine the radius and the interval of convergence. If a power series converges, then it defines a function, and we can say that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n.$$

Differentiations and integrations of $f(x)$ can be found as its derivatives and integrals of the power series, by term-by-term. These functions will be the main point of our focus in this chapter.

Example 1

$\frac{1}{1-x}$ is equal to the power series $1 + x + x^2 + \cdots$ where x is in its interval of convergence $-1 < x < 1$.

4.2 Power Series Solutions

Consider a linear 2nd-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

which can also be written as its standard form

$$y'' + p(x)y' + q(x)y = 0.$$

To see if there exists a power series solution centered at c, we need to see if c is an *ordinary point*.

Definition 4.2.1: Ordinary/Singular Point

A point c is called an **ordinary point** if both $p(x)$ and $q(x)$ are analytic at c. If either $p(x)$ or $q(x)$ is not analytic at c, then c is a **singular point**.

We focus on polynomial coefficients since polynomials are analytic everywhere. Therefore, $p(x)$ and $q(x)$ are analytic everywhere except the points where $a_2(x) = 0$.

Theorem 4.2.1: Existence of a Power Series Solution

If c is an ordinary point, then there exist two linear independent power series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots.$$

which converges on some interval not containing any singular points.

The proof is omitted since it requires complex analysis.

Undetermined Series Coefficient Method

The method for finding a power series solution is similar to the undetermined coefficient method, used in section 3.5. The method is this: substitute $\sum_{n=0}^{\infty} a_n x^n$ to the equation and make a recurrence relation with a_n. This is best explained by an example.

Example 1
Solve $y'' + x^3 y = 0$.

Solution Since 0 is an ordinary point, there exists a power series solution centered at 0

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We substitute this expression to the equation. Since

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots ,$$

$$\begin{aligned} y'' + x^3 y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ &= (2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) + x^3 (a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= 2a_2 + 6a_3 x + 12 a_4 x^2 + \sum_{n=0}^{\infty} \big(a_n + (n+5)(n+4) a_{n+5}\big) x^{n+3} \\ &= 0, \end{aligned}$$

the coefficients of x^i for $i = 1, 2, \ldots$ should be zero. Therefore, we get $a_2 = a_3 = a_4 = 0$, and a recurrence relation

$$a_{n+5} = -\frac{a_n}{(n+5)(n+4)} \quad \text{for } n \geq 0.$$

Since $a_2 = a_3 = a_4 = 0$, $a_i = 0$ if $i \equiv 2, 3, 4 \mod 5$. The solution is

$$y = a_0 + a_1 x - \frac{a_0}{4 \cdot 5} x^5 - \frac{a_1}{5 \cdot 6} x^6 + \frac{a_0}{4 \cdot 5 \cdot 9 \cdot 10} x^{10} + \frac{a_1}{5 \cdot 6 \cdot 10 \cdot 11} x^{11} - \cdots .$$

Here, a_0 and a_1 are coefficients. The equation is linear and 2nd-order, so there exists two linear independent solutions $y_1(x)$ and $y_2(x)$. To make the form $y = a_0 y_1(x) + a_1 y_2(x)$, we group each terms by a_0 and a_1, which gives

$$\begin{aligned} y_1(x) &= 1 - \frac{1}{4 \cdot 5} x^5 + \frac{1}{4 \cdot 5 \cdot 9 \cdot 10} x^{10} - \frac{1}{4 \cdot 5 \cdot 9 \cdot 10 \cdot 14 \cdot 15} x^{15} + \cdots \text{ and} \\ y_2(x) &= x - \frac{1}{5 \cdot 6} x^6 + \frac{1}{5 \cdot 6 \cdot 10 \cdot 11} x^{11} - \frac{1}{5 \cdot 6 \cdot 10 \cdot 11 \cdot 15 \cdot 16} x^{16} + \cdots . \end{aligned}$$

Example 2
Solve $(1+x)y'' - xy' - y = 0$.

Solution The equation has singular points at $x = -1$, and this gives $R = 1$. Since 0 is an ordinary point, there exists a power series solution centered at 0

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

which converges at least for $-1 < x < 1$. We substitute this expression to the equation. Since

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \text{ and}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots,$$

$$\begin{aligned}(1+x)y'' - xy' - y &= (1+x)\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) - x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \sum_{n=0}^{\infty} a_n x^n \\ &= (1+x)(2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) - x(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) \\ &- (a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= 2a_2 - a_0 + \sum_{n=1}^{\infty} \big((n+1)(n+2)a_{n+2} + n(n+1)a_{n+1} - (n+1)a_n\big)x^n \\ &= 0,\end{aligned}$$

the coefficients of x^i for $i = 1, 2, \ldots$ should be zero. Therefore, we get $2a_2 - a_0 = 0$, and a recurrence relation

$$(n+1)(n+2)a_{n+2} + n(n+1)a_{n+1} - (n+1)a_n = 0,$$

$$\text{and } (n+2)a_{n+2} + n a_{n+1} - a_n = 0 \text{ for } n \geq 1.$$

Compared to the example before, since the recurrence relation contains three terms, it is harder to find the general formula for a_i. For this case, we divide into two cases, one assuming $a_1 = 0$, and one assuming $a_0 = 0$.
Case 1: if $\mathbf{a_0 = 0}$ If $a_0 = 0$, $a_2 = 0$. We get

- $3a_3 + a_2 - a_1 = 0$, so $a_3 = \dfrac{-a_2 + a_1}{3} = -\dfrac{1}{3}a_1$
- $4a_4 + 2a_3 - a_2 = 0$, so $a_4 = \dfrac{-6a_3 + 3a_2}{12} = \dfrac{1}{6}a_1$

- $5a_5 + 3a_4 - a_3 = 0$, so $a_5 = \dfrac{-3a_4 + a_3}{5} = -\dfrac{1}{6}a_1$

Case 2: if $\mathbf{a_1 = 0}$ If $a_1 = 0$, we get

- $3a_3 + a_2 - a_1 = 0$, so $a_3 = \dfrac{-a_2 + a_1}{3} = -\dfrac{1}{3}a_2$
- $4a_4 + 2a_3 - a_2 = 0$, so $a_4 = \dfrac{-2a_3 + a_2}{4} = \dfrac{5}{12}a_2$
- $5a_5 + 3a_4 - a_3 = 0$, so $a_5 = \dfrac{-3a_4 + a_3}{5} = -\dfrac{19}{60}a_2$

Therefore, we have two solutions

$$y_1(x) = x - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{6}x^5 + \cdots \text{ and}$$
$$y_2(x) = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{19}{60}x^5 + \cdots.$$

The general solution is

$$\begin{aligned} y &= a_1 y_1 + a_0 y_2 \\ &= a_1\left(x - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{6}x^5 + \cdots\right) + a_2\left(1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 - \frac{19}{60}x^5 + \cdots\right). \end{aligned}$$

We now know how to solve equations with polynomial coefficients, but not all linear differential equations have polynomial coefficients. Still, equations with nonpolynomial coefficients can be solved by using the *Taylor series* to change into polynomial coefficients.

Definition 4.2.2: Taylor Series

The **Taylor series** of a function $f(x)$ at c is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots.$$

If this polynomial exists, then it equals to $f(x)$ near c.

For example, Taylor series of $\sin x$, $\cos x$, and e^x at 0 are

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

$$\cos x = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots, \text{ and}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

Example 3
Solve $y'' - e^x y = 0$.

Solution We will change e^x to a polynomial by using the Taylor series. Since the Taylor series of e^x exists at 0, 0 is an ordinary point. Therefore, there exists a power series solution centered at 0

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

to the equation

$$y'' - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)y = 0.$$

Substituting the power series solution into the equation gives

$$\begin{aligned}
&y'' - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)y \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)\sum_{n=0}^{\infty} a_n x^n \\
&= (2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)(a_0 + a_1 x + a_2 x^2 + \cdots) \\
&= (2a_2 - a_0) + (6a_3 - a_1 - a_0)x + \left(12a_4 - a_2 - a_1 - \frac{1}{2}a_0\right)x^2 \\
&+ \left(20a_5 - a_3 - a_2 - \frac{1}{2}a_1 - \frac{1}{6}a_0\right)x^3 + \cdots \\
&= 0.
\end{aligned}$$

Comparing the coefficients, we get

$$\begin{aligned}
2a_2 - a_0 &= 0, \\
6a_3 - a_1 - a_0 &= 0, \\
12a_4 - a_2 - a_1 - \frac{1}{2}a_0 &= 0, \\
20a_5 - a_3 - a_2 - \frac{1}{2}a_1 - \frac{1}{6}a_0 &= 0,
\end{aligned}$$

and so on. We now divide into two cases, either $a_0 = 0$ or $a_1 = 0$.

Case 1: if $\mathbf{a_0} = \mathbf{0}$ If $a_0 = 0$, $a_2 = 0$. We get

- $6a_3 - a_1 - a_0 = 0$, so $a_3 = \dfrac{a_1 + a_0}{6} = \dfrac{1}{6}a_1$
- $12a_4 - a_2 - a_1 - \dfrac{1}{2}a_0 = 0$, so $a_4 = \dfrac{2a_2 + 2a_1 + a_0}{24} = \dfrac{1}{12}a_1$
- $20a_5 - a_3 - a_2 - \dfrac{1}{2}a_1 - \dfrac{1}{6}a_0 = 0$, so $a_5 = \dfrac{6a_3 + 6a_2 + 3a_1 + a_0}{120} = \dfrac{1}{30}a_1$

Case 2: if $\mathbf{a_1} = \mathbf{0}$ If $a_1 = 0$, we get

- $2a_2 - a_0 = 0$, so $a_2 = \dfrac{1}{2}a_0$
- $6a_3 - a_1 - a_0 = 0$, so $a_3 = \dfrac{a_1 + a_0}{6} = \dfrac{1}{6}a_0$
- $12a_4 - a_2 - a_1 - \dfrac{1}{2}a_0 = 0$, so $a_4 = \dfrac{2a_2 + 2a_1 + a_0}{24} = \dfrac{1}{12}a_0$
- $20a_5 - a_3 - a_2 - \dfrac{1}{2}a_1 - \dfrac{1}{6}a_0 = 0$, so $a_5 = \dfrac{6a_3 + 6a_2 + 3a_1 + a_0}{120} = \dfrac{1}{24}a_0$

Therefore, we have two solutions

$$y_1(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \cdots \text{ and}$$
$$y_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \cdots.$$

The general solution is

$$\begin{aligned} y &= a_1y_1 + a_0y_2 \\ &= a_1\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \cdots\right) + a_2\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \cdots\right). \end{aligned}$$

4.3 The Frobenius Method

The previous section covered power series solutions centered at ordinary points. But what about singular points? There still is a method of finding the power series solution, under some conditions.

Definition 4.3.1: Regular Singular Point

Consider a linear 2nd-order differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where c is a singular point. Then, c is a **regular singular point** if $(x-c)p(x)$ and $(x-c)^2 q(x)$ are both analytic at c.

We also call c irregular if either $(x-c)p(x)$ or $(x-c)^2 q(x)$ is not analytic at c. For example, for any Cauchy-Euler equations of 2nd-order, $x = 0$ is a regular singular point. The **Frobenius method** tells that if c is a regular singular point of a linear 2nd-order differential equation, then one can find a power series solution centered at c.

Theorem 4.3.1: Frobenius Method

If $x = c$ is a regular singular point of a linear 2nd-order differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

then there exists at least one power series solution centered at c, which is

$$\begin{aligned} y &= (x-c)^r \sum_{n=0}^{\infty} a_n (x-c)^n \\ &= a_0(x-c)^r + a_1(x-c)^{r+1} + a_2(x-c)^{r+2} + \cdots . \end{aligned}$$

where r is a constant.

Here, r is a constant to be determined. The method is as follows:

1. Expand $p(x)$ and $q(x)$ to its power series.
2. Substitute $y = (x-c)^r \sum_{n=0}^{\infty} a_n (x-c)^n$.
3. Compare the constant term, which gives the indicial equation.
4. Compare the coefficients for each term and find the solution.

Let $p'(x) = (x-c)p(x)$ and $q'(x) = (x-c)^2 q(x)$. Then, $p'(x)$ and $q'(x)$ are analytic at c. Multiplying the original equation by $(x-c)^2$ gives

$$(x-c)^2 y'' + (x-c)p'(x) + q'(x) = 0.$$

Let $p'(x) = p_0 + p_1(x-c) + p_2(x-c)^2 + \cdots$ and $q'(x) = q_0 + q_1(x-c) + q_2(x-c)^2 + \cdots$. Then, the equation becomes

$$\begin{aligned}(x-c)^2 y'' + (x-c)\big(p_0 + p_1(x-c) + p_2(x-c)^2 + \cdots\big)y'\\ + \big(q_0 + q_1(x) + q_2(x-c)^2 + \cdots\big)y = 0.\end{aligned}$$

Since

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-c)^{n+r-2} \text{ and}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n(x-c)^{n+r-1},$$

substituting gives

$$\begin{aligned}&(x-c)^2 y'' + (x-c)\big(p_0 + p_1(x-c) + p_2(x-c)^2 + \cdots\big)y'\\
&\qquad + \big(q_0 + q_1(x) + q_2(x-c)^2 + \cdots\big)y.\\
&\qquad = (x-c)^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-c)^{n+r-2}\\
&\qquad + (x-c)\big(p_0 + p_1(x-c) + p_2(x-c)^2 + \cdots\big)\left(\sum_{n=0}^{\infty}(n+r)a_n(x-c)^{n+r-1}\right)\\
&\qquad + \big(q_0 + q_1(x) + q_2(x-c)^2 + \cdots\big)\left(\sum_{n=0}^{\infty} a_n(x-c)^{n+r}\right)\\
&\qquad = \big(r(r-1) + p_0 r + q_0\big)(x-c)^r + \cdots = 0.\end{aligned}$$

Therefore, the left-hand side should be zero, and the coefficients of each power of $(x-c)^r$, $(x-c)^{r+1}$, $(x-c)^{r+2}$, ..., should be zero. Comparing the coefficients of these powers of $x-c$ will give you the coefficients of the series solution a_1, a_2, ... For the constant term, the equation

$$r(r-1) + p_0 r + q_0 = 0$$

is called the **indicial equation**, which gives r since p_0 and q_0 are known. Solving the equation, this is now divided into four cases, depending on two roots of r.

Case 1: Distinct Roots not Differing by an Integer
Note that this also includes complex conjugate roots. When the quadratic equation has two distinct real roots r_1 and r_2 not differing by an integer, we have two linear independent solutions

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \text{ and}$$

$$y_2(x) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n = x^{r_1}(b_0 + b_1 x + b_2 x^2 + \cdots).$$

Case 2: Distinct Roots Differing by an Integer
If the two roots satisfy $r_1 - r_2 = n$ where n is a positive integer, the two linear independent solutions are

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \text{ and}$$

$$\begin{aligned} y_2(x) &= c y_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n \\ &= c x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)\ln x + x^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots) \end{aligned}$$

where c is a constant that could be zero.
Case 3: Repeated Real Roots
If two roots are equal, then the two linear independent solutions are

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \text{ and}$$

$$\begin{aligned} y_2(x) &= y_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n \\ &= x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)\ln x + x^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots). \end{aligned}$$

Note that one always can find one solution by substituting $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ into the equation, but in cases 2 and 3, the Frobenius method may fail to find a second solution. If one cannot find a series $\sum_{n=0}^{\infty} b_n x^n$, then the method fails to find a second solution. Still, for some cases, one can try using the reduction order method from section 3.2 to find the second solution. We do not consider the case when the indicial equation has complex conjugate roots.

Example 1
Solve $2x(x-1)y'' - (x+1)y' + y = 0$.

Solution The standard form of the equation is

$$y'' - \frac{x+1}{2x(x-1)}y' + \frac{1}{2x(x-1)} = 0.$$

Since $x = 0$ is a regular singular point of the equation, we try a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n.$$

We have $p(x) = -(x+1)/2x(x-1)$ and $q(x) = 1/2x(x-1)$, so $p'(x) = -(x+1)/2(x-1)$ and $q'(x) = x/2(x-1)$. The Taylor series expansion of $p'(x)$ and $q'(x)$ at 0 are

$$p'(x) = \frac{1}{2} + x + x^2 + x^3 - \cdots \text{ and}$$

$$q'(x) = -\frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^4 - \cdots$$

which gives $p_0 = 1/2$ and $q_0 = 0$. Therefore, the indicial equation is

$$r(r-1) + \frac{1}{2}r = 0,$$

which gives two roots $r_1 = 1/2$ and $r_2 = 0$. These two roots do not differ by an integer. For $r_1 = 1/2$, substituting $y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n$ gives $a_1 = a_2 = a_3 = \cdots = 0$, and hence

$$y_1(x) = \sqrt{x}.$$

For $r_2 = 0$, substituting $y_1(x) = \sum_{n=0}^{\infty} b_n x^n$ gives $b_0 = b_1 = 1$ and $b_2 = b_3 = \cdots = 0$, we have

$$y_2(x) = x + 1.$$

Therefore, the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1\sqrt{x} + c_2(x+1).$$

Example 2

Solve $xy'' + y = 0$.

Solution The standard form of the equation is

$$y'' + \frac{y}{x} = 0,$$

which $x = 0$ is a regular singular point. Since $p'(x) = 0$ and $q'(x) = x$, we have $p_0 = q_0 = 0$, and hence the indicial equation is $r(r-1) = 0$. We have two roots $r = 0, 1$ that differ by an integer. For $r = 1$, substituting $y_1(x) = x\sum_{n=0}^{\infty} a_n x^n$, we get

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots.$$

For the second solution, we look for a solution of the form

$$y_2(x) = cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^n.$$

Substituting $y_2(x)$ to the equation and comparing coefficients, we get

$$y_2(x) = -y_1(x)\ln x + 1 + x + \frac{x^2}{4} - \frac{x^3}{9} + \cdots.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right) + c_2\left(\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right)\ln x \right. \\ &\qquad \left. + 1 + x + \frac{x^2}{4} - \frac{x^3}{9} + \cdots\right). \end{aligned}$$

4.4 Bessel's Equations

Definition 4.4.1: Bessel's Equation

A 2nd-order linear differential equation of the form

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

where v is a constant is called **Bessel's equation**.

Since $x = 0$ is a regular singular point to the equation, we try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Substituting this into the equation, we have

$$\begin{aligned} &x^2y'' + xy' + (x^2 - v^2)y \\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - v^2\sum_{n=0}^{\infty} a_n x^{n+r}. \end{aligned}$$

Then, the indicial equation is

$$r(r-1)+r-v^2=r^2-v^2=0,$$

and $r=\pm v$. Without loss of generality, let $v\geq 0$. We have an equation about a_1 and a recursion formula

$$(r+1)ra_1+(r+1)a_1-v^2a_1=0$$
$$(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}-v^2a_n=0 \text{ for } n=2,3,\ldots.$$

For $r=v$, the equation and the recursion formula becomes

$$(2v+1)a_1=0$$
$$n(n+2v)a_n+a_{n-2}=0.$$

We get $a_1=0$, and thus $a_n=0$ for odd n. So, we consider the recursion formula for only even n and get

$$a_{2n}=-\frac{1}{4n(n+v)}a_{2n-2} \text{ for } n=1,2,\ldots.$$

Solving the recursion formula, we have

$$a_{2n}=\frac{(-1)^n}{2^{2n}n!(1+v)(2+v)\cdots(n+v)}a_0 \text{ for } n=1,2,\ldots.$$

After this, we use the *gamma function.*

Definition 4.4.2: Gamma Function

For any real number r, the **gamma function** is defined by

$$\Gamma(r+1)=\int_0^\infty t^re^{-t}\,dt \text{ for } r>-1.$$

The gamma function is the extension of the factorial to real numbers. The gamma function satisfies the following properties:

- $\Gamma(r+1)=r\Gamma(r)$
- $\Gamma(n+1)=n!$ for $n\in\mathbb{N}$.

Since a_0 could be any number, we let

$$a_0=\frac{1}{2^v\Gamma(1+v)}.$$

Substituting a_0 in the recursion formula gives

$$\begin{aligned} a_{2n} &= \frac{(-1)^n}{2^{2n}n!(1+v)(2+v)\cdots(n+v)}a_0 \\ &= \frac{(-1)^n}{2^{2n}n!(1+v)(2+v)\cdots(n+v)} \cdot \frac{1}{2^v\Gamma(1+v)} \\ &= \frac{(-1)^n}{2^{2n+v}n!\Gamma(1+n+v)} \text{ for } n = 1, 2, \ldots. \end{aligned}$$

The series solution to the equation is called **Bessel's function**.

Definition 4.4.3: Bessel's Function of the First Kind

The series solution to the Bessel's equation is called the **Bessel's function of the first kind** and denoted by

$$J_v(x) = x^v \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+v}n!\Gamma(1+n+v)}x^{2n}.$$

by the same matter, we obtain

$$J_{-v}(x) = x^{-v} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-v}n!\Gamma(1+n-v)}x^{2n}.$$

The functions at least converge on the interval $(0, \infty)$.

Linear Independence when v is not an integer

One question may come out: are $J_v(x)$ and $J_{-v}(x)$ linear independent? For some cases, the answer is yes. If v is not an integer, then the two solutions are linear independent. Before we state the proof directly, we start with some lemmas that help the proof.

Lemma

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proof. We have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{-1/2}e^{-t}\,dt.$$

Using the substitution $t = u^2$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\,du.$$

Squaring both sides gives

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)^2 &= \left(2\int_0^\infty e^{-u^2}\,du\right)^2\\ &= \left(2\int_0^\infty e^{-u^2}\,du\right)\left(2\int_0^\infty e^{-v^2}\,dv\right)\\ &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv.\end{aligned}$$

Using polar coordinates with $u = r\cos\theta$ and $v = r\sin\theta$, we finally get

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)^2 &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv\\ &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta\\ &= 2\pi\int_0^\infty e^{-r^2} r\,dr\\ &= \pi.\end{aligned}$$

Taking the square root of both sides gives

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

■

Lemma

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

Proof. Substituting $v = 1/2$ into the formula of $J_v(x)$ gives

$$\begin{aligned}J_{1/2}(x) &= x^{1/2}\sum_{n=0}^\infty \frac{(-1)^n}{2^{2n+1/2}n!\Gamma(n+3/2)}x^{2n}\\ &= \sqrt{\frac{2}{x}}\sum_{n=0}^\infty \frac{(-1)^n}{2^{2n+1}n!\Gamma(n+3/2)}x^{2n+1}.\end{aligned}$$

The denominator in the fraction can be written as

$$\begin{aligned}2^{2n+1}n!\Gamma(n+3/2) &= 2^{2n+1}n!\cdot(n+1/2)(n-1/2)\cdots(1/2)\Gamma(1/2)\\ &= (2n+1)!\Gamma(1/2)\\ &= (2n+1)!\sqrt{\pi}.\end{aligned}$$

Therefore,

$$\begin{aligned}J_{1/2}(x) &= \sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\sqrt{\pi}}x^{2n+1}\\ &= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\\ &= \sqrt{\frac{2}{\pi x}}\sin x.\end{aligned}$$

Substituting $v=-1/2$ into the formula of $J_v(x)$ gives

$$\begin{aligned}J_{-1/2}(x) &= x^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-1/2}n!\Gamma(n+1/2)}x^{2n}\\ &= \sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!\Gamma(n+1/2)}x^{2n}.\end{aligned}$$

The denominator in the fraction can be written as

$$\begin{aligned}2^{2n}n!\Gamma(n+1/2) &= 2^{2n}n!\cdot(n-1/2)(n-3/2)\cdots(1/2)\Gamma(1/2)\\ &= (2n)!\Gamma(1/2)\\ &= (2n)!\sqrt{\pi}.\end{aligned}$$

Therefore,

$$\begin{aligned}J_{-1/2}(x) &= \sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\sqrt{\pi}}x^{2n}\\ &= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\\ &= \sqrt{\frac{2}{\pi x}}\cos x.\end{aligned}$$

■

Theorem 4.4.1: Linear Independence when v is not an integer

Let $J_v(x)$ and $J_{-v}(x)$ be the Bessel functions with order v and $-v$, respectively. If $v > 0$ is not an integer, then $J_v(x)$ and $J_{-v}(x)$ are linear independent.

Proof. We divide cases into where $v - (-v) = 2v$ is not an integer or an integer. If $2v$ is not an integer, then by case 1 of the Frobenius method, $J_v(x)$ and $J_{-v}(x)$ are linear independent. If $2v$ is an integer, there are two cases: v being an integer or a half-odd integer. We prove that $J_v(x)$ and $J_{-v}(x)$ are linear independent if v is a half-odd integer. If $v = 1/2$, then the two solutions are

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \text{ and } J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Since $\sin x$ and $\cos x$ are linear independent, we have that $J_{1/2}(x)$ and $J_{-1/2}(x)$ are linear independent. For other half-odd integers, one can see that $J_v(x)$ and $J_{-v}(x)$ are linearly independent because the first terms of each function are finite nonzero multiples of x^v and x^{-v}. ■

Therefore, we can conclude that the general solution to the Bessel's equation when v is not an integer is

$$y(x) = c_1 J_v(x) + c_2 J_{-v}(x).$$

Bessel's Equation when $v = 0$

When $v = 0$, Bessel's equation becomes

$$x^2 y'' + xy' + x^2 y = 0.$$

We have one solution $J_0(x)$, but since $v = -v = 0$, we only have one. We need to find a second solution. By the Frobenius method, we know that the second solution will be of the form

$$y_2(x) = J_0(x) \ln x + \sum_{n=1}^{\infty} b_n x^n$$

since the root of the indicial equation is $r_2 = 0$. The derivatives are

$$y_2'(x) = J_0'(x) \ln x + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} n b_n x^{n-1}$$

$$y_2''(x) = J_0''(x) \ln x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=1}^{\infty} n(n-1) b_n x^{n-2}.$$

Substituting the derivatives to the equation gives

$$\begin{aligned}
x^2y_2'' + xy_2' + x^2y_2 &= x^2\big(J_0''(x)\ln x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=1}^{\infty} n(n-1)b_nx^{n-2}\big) \\
&\quad + x\big(J_0'(x)\ln x + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} nb_nx^{n-1}\big) + x^2\big(J_0(x)\ln x + \sum_{n=1}^{\infty} b_nx^n\big) \\
&= \big(x^2J_0''(x) + xJ_0'(x) + x^2J_0(x)\big)\ln x + 2xJ_0'(x) - J_0(x) + J_0(x) \\
&\quad + \sum_{n=1}^{\infty} n(n-1)b_nx^n + \sum_{n=1}^{\infty} nb_nx^n + \sum_{n=1}^{\infty} b_nx^{n+2} \\
&= 2xJ_0'(x) + \sum_{n=1}^{\infty} n(n-1)b_nx^n + \sum_{n=1}^{\infty} nb_nx^n + \sum_{n=1}^{\infty} b_nx^{n+2} = 0,
\end{aligned}$$

and therefore

$$2J_0'(x) + \sum_{n=1}^{\infty} n(n-1)b_nx^{n-1} + \sum_{n=1}^{\infty} nb_nx^{n-1} + \sum_{n=1}^{\infty} b_nx^{n+1} = 0.$$

Since

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}x^{2n},$$

we have

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}n!(n-1)!}x^{2n-1}.$$

This gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-2}n!(n-1)!}x^{2n-1} + \sum_{n=1}^{\infty} n^2b_nx^{n-1} + \sum_{n=1}^{\infty} b_nx^{n+1} = 0.$$

Since the term x^0 only occurs in the second term with coefficient b_1, $b_1 = 0$. Comparing the coefficients of even powers x^{2k}. we get a recursion formula

$$(2k+1)^2b_{2k+1} + b_{2k-1} = 0.$$

Hence $b_n = 0$ for all odd n. Now comparing the coefficients of odd powers x^{2k+1}, we get

$$-1 + 4b_2 = 0$$

$$\frac{(-1)^{k+1}}{2^{2k}(k+1)!k!} + (2k+2)^2b_{2k+2} + b_{2k} = 0$$

and therefore

$$y_2(x) = J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \cdots.$$

Finally, the general solution to the Bessel's equation of order zero is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 J_0(x) + c_2\left(J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \cdots\right). \end{aligned}$$

Bessel's Function of the Second Kind

When v is an integer, we get that $J_v(x)$ and $J_{-v}(x)$ are linear dependent because $J_{-v}(x) = (-1)^v J_v(x)$.

Lemma

$J_{-m}(x) = (-1)^m J_m(x)$ for a positive integer m.

Proof. We have

$$\begin{aligned} J_{-m}(x) &= x^{-m}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-m}n!\Gamma(1+n-m)}x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n-m}n!(n-m)!}x^{2n-m} \\ &= \sum_{n=m}^{\infty}\frac{(-1)^n}{2^{2n-m}n!(n-m)!}x^{2n-m} \end{aligned}$$

since $\Gamma(r)$ is infinite when $r < -1$, and therefore the value of the terms becomes 0 when $n < m$. With the substitution $n = m + k$, we get

$$\begin{aligned} \sum_{n=m}^{\infty}\frac{(-1)^n}{2^{2n-m}n!(n-m)!}x^{2n-m} &= \sum_{k=0}^{\infty}\frac{(-1)^{m+k}}{2^{2k+m}k!(m+k)!}x^{2k+m} \\ &= (-1)^m\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+m}k!(m+k)!}x^{2k+m} \\ &= (-1)^m J_m(x). \end{aligned}$$

Therefore, we have the result $J_{-m}(x) = (-1)^m J_m(x)$ for a positive integer m. ■

To find a second series solution when v is an integer, we define a new function, called the *Bessel's function of the second kind*.

Definition 4.4.4: Bessel's Function of the Second Kind

If v is not an integer, the function $Y_v(x)$ defined by

$$Y_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}$$

is called the **Bessel's function of the second kind**. We also define

$$Y_n(x) = \lim_{v\to n} Y_v(x),$$

by L'Hôpital's rule, where n is an integer.

Then, it can be proved that $Y_v(x)$ is another solution to Bessel's equation that is linear independent to $J_v(x)$. We therefore conclude that the general solution to the Bessel's equation is

$$y(x) = c_1 J_v(x) + c_2 Y_v(x).$$

4.5 Legendre's Equations

Definition 4.5.1: Legendre's Equation

A 2nd-order linear differential equation of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where n is a constant is called **Legendre's equation**.

Since $x = 0$ is not a singular point to the equation, we try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Substituting the series into the equation gives

$$\begin{aligned}
&(1-x^2)y'' - 2xy' + n(n+1)y \\
&\qquad = (1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k \\
&\qquad = \big(n(n+1)a_0 + 2a_2\big) + \big((n-1)(n+2)a_1 + 6a_3\big)x \\
&\qquad + \sum_{i=2}^{\infty} \big((i+1)(i+2)a_{i+2} + (n-i)(n+i+1)a_i\big)x^i = 0.
\end{aligned}$$

We then obtain a recursion formula

$$a_2 = -\frac{n(n+1)}{2}a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{6}a_1$$

$$a_{i+2} = -\frac{(n-i)(n+i+1)}{(i+1)(i+2)}a_i.$$

Therefore, the general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} - \cdots \text{ and}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots.$$

The Solution when $n = 0$ and $n = 1$

Notice that we have $y_1(x) = 1$ when $n = 0$. Also, $y_2(x)$ becomes

$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots.$$

With the Taylor series expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots,$$

We get $y_2(x) = \big(\ln(1+x) - \ln(1-x)\big)/2$ because

$$\frac{1}{2}\big(\ln(1+x) - \ln(1-x)\big) = \frac{1}{2}\left(\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right)\right)$$

$$= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 = y_2(x).$$

Therefore, the general solution to the Legendre's equation when $n = 0$ is

$$y(x) = a_0 + a_1' \ln\frac{1+x}{1-x},$$

where $a_1' = a_1/2$.

When $n = 1$, we have $y_2(x) = x$, and

$$y_1(x) = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 = 1 - \frac{1}{2}x \ln \frac{1+x}{1-x}.$$

The general solution to the Legendre's equation when $n = 1$ is

$$y(x) = a_0\left(1 - \frac{1}{2}x \ln \frac{1+x}{1-x}\right) + a_1 x.$$

Legendre Polynomials

Notice when n is an even integer, then $y_1(x)$ terminates, and when n is an odd integer, then $y_2(x)$ terminates. These polynomials are called *Legendre polynomials*.

Definition 4.5.2: Legendre Polynomials

If n is an integer, then the n-th degree polynomial $P_n(x)$ obtained by terminated $y_1(x)$ or $y_2(x)$ is called the **Legendre polynomial**.

The first few Legendre polynomials are:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

These polynomials satisfy the Legendre's equation for $n = 0, 1, \ldots, 5$, respectively.

Chapter 5

Laplace Transforms

This chapter covers Laplace transforms that are used to solve ordinary differential equations. Laplace transform is a useful technique in solving ordinary differential equations. We first start with the definition.

5.1 Definition of the Laplace Transform

Definition 5.1.1: Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

is said to be the **Laplace transform** of f provided the integral converges.

We usually use the notation

$$\mathscr{L}\{f(t)\} = F(s),\ \mathscr{L}\{g(t)\} = G(s), \text{ and } \mathscr{L}\{y(t)\} = Y(s).$$

Example 1

Evaluate $\mathscr{L}\{1\}$.

Solution

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt = \left.\frac{-e^{-st}}{s}\right|_0^\infty = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

provided $s > 0$. If $s < 0$, the integral diverges.

Example 2

Evaluate $\mathscr{L}\{e^{at}\}$, where a is any real number.

Solution

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \left.\frac{e^{(-s+a)t}}{-s+a}\right|_0^\infty = \lim_{b\to\infty} \frac{e^{(-s+a)b}-1}{-s+a} = \frac{1}{s-a}$$

provided $s > a$. If $s < a$, the integral diverges.

The notation $\int_0^\infty f(t)\,dt$ is usually used for $\lim_{b\to\infty} \int_0^b f(t)\,dt$. Also, assume the conditions for s are satisfied.

Theorem 5.1.1: Linearity of the Laplace Transform

Suppose that there exists $\mathscr{L}\{f_1\}$ and $\mathscr{L}\{f_2\}$ for $s > a_1$ and $s > a_2$. Then, for $s > max\{a_1, a_2\}$,

$$\mathscr{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}..$$

Proof.

$$\begin{aligned}\mathscr{L}\{c_1 f_1 + c_2 f_2\} &= \int_0^\infty e^{-st}\big(c_1 f_1(t) + c_2 f_2(t)\big)\, dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\ &= c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}..\end{aligned}$$ ■

Some transforms of basic functions are:

$$\mathscr{L}\{1\} = \frac{1}{s}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1, 2, 3\ldots, \quad \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathscr{L}\{\sin kt\} = \frac{k}{s^2 + k^2}, \quad \mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

Example 3

Evaluate $\mathscr{L}\{t - t^2 + 2e^{4t}\}$.

Solution

$$\mathscr{L}\{t - t^2 + 2e^{4t}\} = \mathscr{L}\{t\} - \mathscr{L}\{t^2\} + 2\mathscr{L}\{e^{4t}\} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s-4}.$$

Existence and Uniqueness

Of course, the improper integral $\int_0^\infty e^{-st} f(t)\, dt$ might not exist. Then, when does the Laplace transform exist? We propose a theorem of a condition for existence. We first define two terminologies, *piecewise continuous* and *exponential order*.

Definition 5.1.2: Piecewise Continuous Function

A function f is **piecewise continuous** when the number of discontinuous points in $(-\infty, \infty)$ are finite, and the function doesn't have a divergent limit.

Definition 5.1.3: Exponential Order

A function f is of **exponential order** when there exists constants a, $k > 0$ and $T > 0$ such that
$$f(t) \leq ke^{at} \text{ when } t > T.$$

This means that f should be eventually smaller than an exponential function. For example, $f(t) = t^n$ is of exponential order for any natural number n, but $f(t) = e^{t^2}$ is not of exponential order.

Theorem 5.1.2: Sufficient Condition for the Existence of Laplace Transform

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order. Then the Laplace transform of f exists for $s > 0$.

Proof. We divide $[0, \infty)$ to $[0, T)$ and $[T, \infty)$.

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt.$$

We get that $\int_0^T e^{-st} f(t)\ dt$ is finite. Since f is of exponential order, there exists some constants a, $k > 0$ and $T > 0$ such that

$$|f(t)| \leq ke^{at} \text{ for } t > T.$$

Therefore,

$$\begin{aligned}\left| \int_T^\infty e^{-st} f(t)\ dt \right| &\leq \int_T^\infty |e^{-st} f(t)|\ dt \\ &\leq k \cdot \int_0^\infty e^{-st} \cdot e^{at}\ dt \\ &= k \cdot \frac{e^{-(s-a)T}}{s-a} \text{ for } s > a.\end{aligned}$$ ■

We now know about existence, but how about uniqueness? What if there are two different Laplace transforms for a function? That is actually not the case, and Laplace transform is unique. However, the proof of uniqueness is beyond this level, so we do not state it here. From now on, one can assume that Laplace transform of a function is unique.

Theorem 5.1.3: Uniqueness of the Laplace Transform

Assume that f, $g : [0, \infty) \to \mathbf{R}$ are continuous and of exponential order. If $\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\}$, the $f(t) = g(t)$.

5.2 The Inverse Laplace Transform

Definition 5.2.1: Inverse Laplace Transform

If $F(s) = \mathscr{L}\{f(t)\}$ we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

Unlike the Laplace transform where there is a given formula, inverse transforms don't have a specific formula, and it only could be found by knowing Laplace transforms of functions. This means that inverse transforms of arbitrary functions cannot be calculated. It is specifically shown in example 1 of this section.

Inverse transforms of some functions are:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n,\ n = 1, 2, 3\ldots, \quad \mathscr{L}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathscr{L}\left\{\frac{k}{s^2+k^2}\right\} = \sin\, kt, \quad \mathscr{L}\left\{\frac{s}{s^2+k^2}\right\} = \cos\, kt$$

Like the Laplace transform, the inverse transform is also linear.

Theorem 5.2.1: Linearity of the Inverse Transform

The inverse Laplace transform is a linear transform. That is, for constants c_1 and c_2,

$$\mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\}.$$

Example 1

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3.$$

Example 2

Evaluate $\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+9}\right\}$.

Solution

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+9}\right\} &= \mathscr{L}^{-1}\left\{2\cdot\frac{s}{s^2+9}+\frac{3}{s^2+9}\right\} \\
&= 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\
&= 2\cos 3t+\sin 3t.
\end{aligned}
$$

5.3 Transforms of Derivatives and Integrals

In this section, we see some properties of Laplace transforms and how they can be used to solve ordinary differential equations.

Theorem 5.3.1: Transforms of Derivatives

Assume that f' is piecewise continuous on $[0,\infty)$ and f is of exponential order. Then,

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

If $f, f', \cdots, f^{(n-1)}$ are continuous on $[0,\infty)$, f is of exponential order, and $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Proof. We use induction. For $n=1$,

$$
\begin{aligned}
\mathscr{L}\{f'(t)\} &= \int_0^\infty e^{-st}f'(t)\,dt \\
&= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt \\
&= -f(0)+s\mathscr{L}\{f(t)\} \\
&= sF(s)-f(0).
\end{aligned}
$$

Assume the equation holds for $n=k$. So,

$$\mathscr{L}\{f^{(k)}(t)\} = s^kF(s) - s^{k-1}f(0) - s^{k-2}f'(0) - \cdots - f^{(k-1)}(0).$$

For $n = k+1$,

$$\begin{aligned}
\mathscr{L}\{f^{(k+1)}(t)\} &= \int_0^\infty e^{-st} f^{(k+1)}(t)\,dt \\
&= e^{-st} f^{(k)}(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(k)}(t)\,dt \\
&= -f^{(k)}(0) + s\mathscr{L}\{f^{(k)}(t)\} \\
&= s(s^k F(s) - s^{k-1}f(0) - s^{k-2}f'(0) - \cdots - f^{(k-1)}(0)) - f^{(k)}(0) \\
&= s^{k+1}F(s) - s^k f(0) - s^{k-1}f'(0) - \cdots - f^{(k)}(0),
\end{aligned}$$

which completes the induction. ■

Theorem 5.3.2: Transforms of Integrals

If f is piecewise continuous on $[0,\infty)$ and of exponential order, then

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}, \text{ and}$$

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau.$$

Proof. Let $g(t) = \int_0^t f(\tau)\,d\tau$. We first prove that $g(t)$ is of exponential order. Since f is of exponential order, there exists k, a and τ such that $|f(t)| \le ke^{at}$. Then,

$$|g(t)| = \left|\int_0^t f(\tau)\,d\tau\right| \le \int_0^t |f(\tau)|\,d\tau \le \int_0^t ke^{a\tau}\,d\tau = \frac{k}{a}(e^{at}-1) < \frac{k}{a}e^{at},$$

which shows that g is also of exponential order. Also, since $\frac{d}{dt}g(t) = f(t)$, and $g(0) = 0$, by the Transforms of Derivatives theorem,

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{d}{dt}g(t)\right\} = s\mathscr{L}\{g(t)\}(s) - g(0) = s\mathscr{L}\{g(t)\}.$$

Dividing by s for both sides gives us

$$\mathscr{L}\{g(t)\} = \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}.$$ ■

Example 1

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}$.

Solution

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}\\ &= \int_0^t sin\tau\, d\tau\\ &= 1 - cost.\end{aligned}$$

Example 2

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+1)}\right\}\\ &= \int_0^t (1 - cos\tau)\, d\tau\\ &= t - sint.\end{aligned}$$

Solving Differential Equations with Laplace Transforms

Laplace transforms can be used in solving ordinary differential equations, especially initial-value problems. The steps for solving initial-value problems are:

1. Apply the Laplace transform for both sides of the initial-value problem, which changes $y(t)$ to $Y(s)$.
2. Solve the equation with respect to $Y(s)$.
3. Apply the inverse transform to the solution of $Y(s)$, and you get the solution $y(t)$ to the initial-value problem.

Example 3

Solve $y' + y = 2cost$, $y(0) = 1$.

Solution Applying Laplace transform to both sides gives you

$$\begin{aligned}sY(s) - y(0) + Y(s) &= 2\cdot\frac{s}{s^2+1}\\ (s+1)Y(s) - 1 &= 2\cdot\frac{s}{s^2+1}.\end{aligned}$$

If you solve for $Y(s)$, you get

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{2s}{(s+1)(s^2+1)} \\ &= \frac{s}{s^2+1} + \frac{1}{s^2+1}. \end{aligned}$$

If you apply the inverse transform for both sides, you finally obtain

$$y(t) = \cos t + \sin t$$

which is the solution of the equation given.

5.4 Translation Theorems

First Translation Theorem

Now you know Laplace transforms of some basic functions, but what about products of basic functions? For example, how would you calculate $\mathscr{L}\{e^{3t}\sin t\}$ or $\mathscr{L}\{e^{-2t}t^4\}$? The **first translation theorem** helps to find the Laplace transform of a function multiplied by an exponential function.

Theorem 5.4.1: First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\begin{aligned} \mathscr{L}\{e^{at}f(t)\} &= F(s-a), \text{and} \\ \mathscr{L}^{-1}\{F(s-a)\} &= e^{at}f(t). \end{aligned}$$

We also use the notation $F(s)\Big|_{s\to s-a}$ for $F(s-a)$.

Proof.

$$\begin{aligned} \mathscr{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-(s-a)t}f(t)\ dt \\ &= F(s-a). \end{aligned}$$ ∎

Example 1

Evaluate $\mathscr{L}\{e^{2t}t^5\}$.

Solution

$$\mathscr{L}\{e^{2t}t^5\} = \mathscr{L}\{t^5\}_{s\to s-2} = \left.\frac{5!}{s^6}\right|_{s\to s-2} = \frac{120}{(s-2)^6}.$$

Example 2

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2-4s+13}\right\}$.

Solution

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{s^2-4s+13}\right\} &= \mathscr{L}^{-1}\left\{\left.\frac{s+2}{s^2+9}\right|_{s\to s-2}\right\}\\ &= \mathscr{L}^{-1}\left\{\left.\left(\frac{s}{s^2+9}+\frac{2}{3}\cdot\frac{3}{s^2+9}\right)\right|_{s\to s-2}\right\}\\ &= e^{2t}\cos 3t + \frac{2}{3}e^{2t}\sin 3t.\end{aligned}$$

Example 3

Solve $y'' - 2y' + 1y = te^t$, $y(0) = 0$, $y'(0) = 4$.

Solution Applying Laplace transform to both sides gives you

$$\mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} + \mathscr{L}\{y\} = \mathscr{L}\{te^t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2Y(s) + 2y(0) + Y(s) = \frac{1}{(s-1)^2}$$

If you solve for $Y(s)$, you get

$$\begin{aligned}(s^2-2s+1)Y(s) &= 4 + \frac{1}{(s-1)^2}\\ Y(s) &= \frac{4}{(s-1)^2} + \frac{1}{(s-1)^4}.\end{aligned}$$

If you apply the inverse transform for both sides, you finally obtain

$$y(t) = 4te^t + \frac{1}{6}t^3e^t.$$

Second Translation Theorem

Differential equations we have encountered until now could also be solved by the variation of parameter method. However, Laplace transforms are used frequently when some functions are special: functions that are not continuous. The **second**

translation theorem is used to solve differential equations involving discontinuous functions.

Definition 5.4.1: Unit Step Function

The **unit step function** $\mathscr{U}(t-a)$ is defined as

$$\mathscr{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

Another name for this function is the *heaviside function.* However, we will call it as unit step function here.

When a function is multiplied by $\mathscr{U}(t-a)$, the function becomes 0 for $t < a$, and itself for $t \geq a$. That is,

$$f(t)\mathscr{U}(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a. \end{cases}$$

If you want to shift the function a units to the right, you can take

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a. \end{cases}$$

Also, general piecewise functions of the type

$$f(t) = \begin{cases} g(t) & t < a \\ h(t) & t \geq a. \end{cases}$$

can be expressed as

$$f(t) = g(t) - \big(g(t) - h(t)\big)\mathscr{U}(t-a).$$

Similarly, piecewise functions of three cases

$$f(t) = \begin{cases} g(t) & t < a \\ h(t) & a \leq t < b \\ g(t) & t \geq b \end{cases}$$

can be written

$$f(t) = g(t) + \big(h(t) - g(t)\big)\big[\mathscr{U}(t-a) - \mathscr{U}(t-b)\big].$$

These expressions of unit step functions can be generalized to functions of several cases, even more than three.

Theorem 5.4.2: Second Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s), \text{ and}$$
$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

This can also be written as

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

with $g(t) = f(t-a)$.

Proof.

$$\begin{aligned}\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} &= \int_0^a e^{-st}f(t-a)\mathscr{U}(t-a)\ dt + \int_a^\infty e^{-st}f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_0^a 0\ dt + \int_a^\infty e^{-st}f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_a^\infty e^{-st}f(t-a)\mathscr{U}(t-a)\ dt.\end{aligned}$$

Substituting $v = t - a$ gives $dv = dt$, and

$$\begin{aligned}\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} &= \int_a^\infty e^{-st}f(t-a)\mathscr{U}(t-a)\ dt \\ &= \int_a^\infty e^{-s(v+a)}f(v)\ dv \\ &= e^{-as}\int_a^\infty e^{-sv}f(v)\ dv \\ &= e^{-as}\mathscr{L}\{f(t)\}.\end{aligned}$$

■

Corollary : Laplace Transform of a Unit Step Function

$$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\left\{\frac{1}{s}\right\} = \frac{e^{-as}}{s}.$$

Example 4

Evaluate $\mathscr{L}\{\cos t\mathscr{U}(t-\pi)\}$.

Solution

$$\mathscr{L}\{\cos t\mathscr{U}(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\cos(t+\pi)\} = -e^{\pi s}\mathscr{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}.$$

Example 5

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{s-2}e^{-6s}\right\}$.

Solution

$$\mathscr{L}^{-1}\left\{\frac{1}{s-2}e^{-6s}\right\} = e^{2(t-6)}\mathscr{U}(t-6).$$

Example 6

Solve $y' - 2y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin t & t \ge \pi. \end{cases}$

Solution $f(t)$ can be written as $f(t) = \sin t\,\mathscr{U}(t-\pi)$.

Applying Laplace transform to both sides gives you

$$\begin{aligned}\mathscr{L}\{y'\} - 2\mathscr{L}\{y'\} &= \mathscr{L}\{\sin t\,\mathscr{U}(t-\pi)\} \\ sY(s) - y(0) - 2Y(s) &= \frac{1}{s^2+1}e^{-\pi s}\end{aligned}$$

If you solve for $Y(s)$, you get

$$\begin{aligned}(s-2)Y(s) &= -\frac{1}{s^2+1}e^{-\pi s} \\ Y(s) &= -\frac{1}{(s-2)(s^2+1)} \\ &= \frac{1}{5}\cdot\frac{s+2}{s^2+1} - \frac{1}{5}\cdot\frac{1}{s-2}\end{aligned}$$

If you apply the inverse transform for both sides, you finally obtain

$$\begin{aligned}y(t) &= \frac{1}{5}\cos(t-\pi)\mathscr{U}(t-\pi) + \frac{2}{5}\sin(t-\pi)\mathscr{U}(t-\pi) - \frac{1}{5}e^{2(t-\pi)}\mathscr{U}(t-\pi) \\ &= \begin{cases} 0 & t < \pi \\ \frac{1}{5}\cos(t-\pi) + \frac{2}{5}\sin(t-\pi) - \frac{1}{5}e^{2(t-\pi)} & t \ge \pi. \end{cases}\end{aligned}$$

The solution of a differential equation including unit step functions may not be differentiable at some points. In this case, we differentiate piecewise, so that the function is continuous, and each part of the function satisfies the differential equation. For the example above, each side of the solution satisfies the differential equation. Also, the solution is continuous because $\lim_{t\to\pi} y(t) = 0 = y(\pi)$.

5.5 Derivatives and Integrals of Transforms

More properties are stated to make evaluating Laplace transforms easier.

Theorem 5.5.1: Derivatives of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, \cdots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), \text{ and}$$

$$\mathscr{L}^{-1}\left\{\frac{d^n}{ds^n}\right\} = (-1)^n t^n f(t).$$

Proof. We use induction. For $n = 1$, since

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\ dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\big(e^{-st} f(t)\big)\ dt \text{ (by Leibniz Rule)} \\ &= \int_0^\infty -e^{-st}\cdot t f(t)\ dt = -\mathscr{L}\{tf(t)\}, \text{ and}\end{aligned}$$

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}.$$

Assume the equation holds for $n = k$. So,

$$\mathscr{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k} F(s).$$

For $n = k + 1$,

$$\begin{aligned}\frac{d}{ds}\left((-1)^k \frac{d^k}{ds^k}F(s)\right) &= (-1)^k \frac{d^{k+1}}{ds^{k+1}} F(s) \\ &= \frac{d}{ds}\int_0^\infty e^{-st} t^k f(t)\ dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\big(e^{-st} t^k f(t)\big)\ dt \text{ (by Leibniz Rule)} \\ &= \int_0^\infty -e^{-st}\cdot t \cdot t^k f(t)\ dt = -\mathscr{L}\{t^{k+1} f(t)\}.\end{aligned}$$

Therefore,

$$\mathscr{L}\{t^{k+1} f(t)\} = (-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}},$$

which completes the induction. ■

The Leibniz Rule used in the proof is a theorem that interchanges the derivative operator with the partial derivative operator inside the integral.

$$\frac{d}{ds}\left(\int_a^b f(x,t)\ dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\ dt.$$

Theorem 5.5.2: Integrals of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(r)\,dr, \text{ and}$$

$$\mathscr{L}^{-1}\left\{\int_s^\infty F(r)\,dr\right\} = \frac{f(t)}{t}.$$

Proof.

$$\begin{aligned}
\int_s^\infty F(r)\,dr &= \int_s^\infty \left(\int_0^\infty e^{-rt} f(t)\,dt\right) dr \\
&= \int_0^\infty \left(\int_s^\infty e^{-rt} f(t)\,dr\right) dt \text{ (Changing the order of integration)} \\
&= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt \\
&= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt \\
&= \mathscr{L}\left\{\frac{f(t)}{t}\right\}.
\end{aligned}$$

■

Example 1

Evaluate $\mathscr{L}^{-1}\left\{\ln\frac{s+3}{s-2}\right\}$.

Solution

$$\begin{aligned}
\text{Since } \frac{d}{ds}\left(\ln\frac{s+3}{s-2}\right) &= \frac{d}{ds}\big(\ln(s+3) - \ln(s-2)\big) \\
&= \frac{1}{s+3} - \frac{1}{s-2} = \mathscr{L}\{-tf(t)\},
\end{aligned}$$

$$-tf(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+3} - \frac{1}{s-2}\right\}$$
$$= e^{-3t} - e^{2t},$$
$$\text{and } f(t) = \frac{e^{2t} - e^{-3t}}{t}.$$

Example 2

Solve $y'' + y = te^t$, $y(0) = 0$, $y'(0) = 1$.

Solution Apply Laplace transforms to both sides, we get

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = \mathscr{L}\{te^t\}$$
$$s^2Y(s) - sy(0) - y'(0) + Y(s) = -\frac{d}{ds}\frac{1}{s-1}$$

Solve for $Y(s)$, then

$$(s^2+1)Y(s) = 1 + \frac{1}{(s-1)^2}$$
$$Y(s) = \frac{1}{s^2+1} + \frac{1}{(s^2+1)(s-1)^2}$$
$$= \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{1}{2}\frac{1}{s-1}$$

Finally, applying the inverse transform to both sides gives you the solution

$$y(t) = \frac{1}{2}\cos t + \sin t + \frac{1}{2}te^t - \frac{1}{2}e^t.$$

5.6 Convolution

Is the Laplace transform multiplicative? The answer is no, and $\mathscr{L}\{fg\} \neq \mathscr{L}\{f\}\mathscr{L}\{g\}$. Instead, an operation called *convolution* is developed to multiply two transforms.

Definition 5.6.1: Convolution

If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the **convolution** of f and g, denoted $f * g$, is a function defined by

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Example 1

Evaluate $t * \sin t$.

Solution

$$\begin{aligned} e^t * t &= \int_0^t e^\tau \cdot (t-\tau)\, d\tau \\ &= \int_0^t \left(te^\tau - \tau e^\tau\right) d\tau \\ &= te^t - t - te^t + e^t - 1 = e^t - t - 1. \end{aligned}$$

As written in the beginning of the text, the usage of convolution arises when multiplying two Laplace transforms. The theorem is called the **convolution theorem**.

Theorem 5.6.1: Convolution Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s), \text{ and}$$
$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g.$$

Proof.

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \text{ and } G(s) = \mathscr{L}\{g(t)\} = \int_0^\infty e^{-s\gamma} g(\gamma)\, d\gamma.$$

Then,

$$\begin{aligned} F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right)\left(\int_0^\infty e^{-s\gamma} g(\gamma)\, d\gamma\right) \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+\gamma)} f(\tau) g(\gamma)\, d\tau\, d\gamma \\ &= \int_0^\infty f(\tau)\, d\tau \int_0^\infty e^{-s(\tau+\gamma)} g(\gamma)\, d\gamma \end{aligned}$$

If we let $t = \tau + \gamma$, since $dt = d\gamma$, so

$$F(s)G(s) = \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty e^{-st} g(t-\tau)\, dt.$$

Because f and g are piecewise continuous on $[0, \infty)$ and of exponential order, we can change the order of integration. Therefore,

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-st}\, dt \int_0^t f(\tau)g(t-\tau)\, d\tau \\ &= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\, d\tau \right) dt \\ &= \mathscr{L}\{f * g\}. \end{aligned}$$

■

Corollary : Transforms of Integrals

$$\mathscr{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{F(s)}{s}.$$

Proof.

$$\begin{aligned} \mathscr{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} &= \mathscr{L}\left\{ \int_0^t f(\tau) \cdot 1\, d\tau \right\} \\ &= \mathscr{L}\{f(t) * 1\} \\ &= \mathscr{L}\{f(t)\} \cdot \mathscr{L}\{1\} \\ &= F(s) \cdot \frac{1}{s} \\ &= \frac{F(s)}{s}. \end{aligned}$$

■

Example 2

Evaluate $\mathscr{L}\left\{ \int_0^t \sin\tau \cos(t-\tau)\, d\tau \right\}$.

Solution

$$\begin{aligned} \mathscr{L}\left\{ \int_0^t \cos\tau \sin(t-\tau)\, d\tau \right\} &= \mathscr{L}\{\cos t * \sin t\} \\ &= \mathscr{L}\{\cos t\} \cdot \mathscr{L}\{\sin t\} \\ &= \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}. \end{aligned}$$

Example 3

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$.

Solution

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\cdot\frac{1}{s^2+k^2}\right\} \\
&= \frac{1}{k^2}\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\cdot\frac{k}{s^2+k^2}\right\} \\
&= \frac{1}{k^2}(\sin t * \sin t) \\
&= \frac{1}{k^2}\int_0^t \sin k\tau \;\, \sin k(t-\tau)\; d\tau \\
&= \frac{1}{k^2}\int_0^t \frac{1}{2}\big(\cos k(2\tau-t) - \cos kt\big)\; d\tau \\
&= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau-t) - \tau\cos kt\right]_0^t \\
&= \frac{\sin kt - kt\cos kt}{2k^3}.
\end{aligned}$$

Properties of Convolution

Convolution has the following properties:

- The associative property, i.e. $f*(g*h) = (f*g)*h$
- The commutative property, i.e. $f*g = g*f$
- The distributive property, i.e. $f*(g+h) = f*g + f*h$
- $f*0 = 0*f = 0$.

Integral Equations

There are not only differential equations, but also integral equations! *Integral equations* are simply functional equations that contain integrals. Solving integral equations are very similar to solving differential equations. Especially, the convolution theorem is used frequently while solving integral equations. There are also equations that contain both derivatives and integrals. Such equations are called *integrodifferential equations.*

Example 4

Solve $y(t) + \int_0^t y(\tau)e^{t-\tau} = 3t^2$.

Solution First, we apply the Laplace transform for both sides.

$$\mathscr{L}\{y(t)\} + \mathscr{L}\left\{\int_0^t y(\tau)e^{t-\tau}\right\} = \mathscr{L}\{3t^2\}$$

$$\mathscr{L}\{y(t)\} + \mathscr{L}\{y(t) * e^t\} = \mathscr{L}\{3t^2\}$$

$$\mathscr{L}\{y(t)\} + \mathscr{L}\{y(t)\} \cdot \mathscr{L}\{e^t\} = \mathscr{L}\{3t^2\}$$

$$Y(s) + \frac{1}{s-1}Y(s) = \frac{6}{s^3}$$

Then, solving for $Y(s)$ gives

$$\frac{s}{s-1}Y(s) = \frac{6}{s^3}$$

$$Y(s) = \frac{6s-6}{s^4}$$

$$= \frac{6}{s^3} - \frac{6}{s^4} = 3\frac{2}{s^3} - \frac{6}{s^4}.$$

Therefore, if you apply the inverse transform, you get the solution

$$y(t) = 3t^2 - t^3.$$

5.7 The Dirac Delta Function

Definition 5.7.1: Unit Impulse

The **unit impulse function** $\delta_a(t-t_0)$ is defined as

$$\delta_a(t-t_0) = \begin{cases} 0 & t < t_0 - a \\ \dfrac{1}{2a} & t_0 - a \leq t < t_0 + a \\ 0 & t \geq t_0 + a \end{cases}$$

where $a > 0$ and $t_0 > 0$.

The unit impulse function has the following property:

$$\int_0^\infty \delta_a(t-t_0) = 1.$$

Definition 5.7.2: Dirac Delta Function

The **Dirac delta function** $\delta(t-t_0)$ is defined by the limit

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0).$$

The Dirac delta function has the following properties:

- $\delta(t-t_0) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0, \end{cases}$ and
- $\int_0^\infty \delta(t-t_0)\ dt = 1.$

For usual functions, $\int_0^\infty \delta(t-t_0)\ dt = 0$, but actually $\int_0^\infty \delta(t-t_0)\ dt = 1$. This is because the Dirac delta function is not actually a function–it is a distribution. The Dirac delta function doesn't contain any meaning itself, but it is characterized with other functions during integration.

Theorem 5.7.1: Shifting Property of Dirac Delta Function

If f is a continuous function, then

$$\int_0^\infty \delta(t-t_0) f(t)\ dt = f(t_0).$$

Proof.

$$\begin{aligned}\int_0^\infty \delta(t-t_0) f(t)\ dt &= \lim_{a\to 0} \int_0^\infty \delta_a(t-t_0) f(t)\ dt \\ &= \lim_{a\to 0} \frac{1}{2a} \int_{t_0-a}^{t_0+a} f(t)\ dt\end{aligned}$$

By the mean value theorem for integrals, there exists $\tilde{t} \in (t_0-a, t_0+a)$ such that

$$\int_{t_0-a}^{t_0+a} f(t)\ dt = 2af(\tilde{t}).$$

Finally,

$$\begin{aligned}\int_0^\infty \delta(t-t_0) f(t)\ dt &= \lim_{a\to 0} \frac{1}{2a} \int_{t_0-a}^{t_0+a} f(t)\ dt \\ &= \lim_{a\to 0} \frac{1}{2a} \big(2af(\tilde{t})\big) \\ &= f(t_0)\end{aligned}$$

Since $\tilde{t} \to 0$ as $a \to \infty$. ∎

Theorem 5.7.2: Transform of the Dirac Delta Function

For $t_0 > 0$, $\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$.

There are two proofs, using the shifting property or the unit step function. Both proofs are stated.

Proof. If we set $f(t) = e^{-st}$, then

$$\mathscr{L}\{\delta(t-t_0)\} = \int_0^\infty \delta(t-t_0)\cdot e^{-st}\; dt = e^{-st_0}$$

Since $f(t_0) = e^{-st_0}$. ∎

Proof. We first write the Dirac delta function as a combination of unit step functions.

$$\delta_a(t-t_0) = \frac{1}{2a}\Big(\mathscr{U}\big(t-(t_0-a)\big) - \mathscr{U}\big(t-(t_0+a)\big)\Big).$$

If we apply the Laplace transform,

$$\begin{aligned}\mathscr{L}\{\delta_a(t-t_0)\} &= \mathscr{L}\left\{\frac{1}{2a}\Big(\mathscr{U}\big(t-(t_0-a)\big) - \mathscr{U}\big(t-(t_0+a)\big)\Big)\right\} \\ &= \frac{1}{2a}\left(\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right) \\ &= e^{-st_0}\left(\frac{e^{as}-e^{-as}}{2as}\right).\end{aligned}$$

Since the Dirac delta function is the unit impulse when $a \to 0$,

$$\begin{aligned}\mathscr{L}\{\delta(t-t_0)\} &= \lim_{a\to 0}\mathscr{L}\{\delta_a(t-t_0)\} \\ &= e^{-st_0}\lim_{a\to 0}\left(\frac{e^{as}-e^{-as}}{2as}\right) \\ &= e^{-st_0}\lim_{a\to 0}\left(\frac{se^{as}+se^{-as}}{2as}\right) \quad \text{(by L'Hôpital's Rule)} \\ &= e^{-st_0}.\end{aligned}$$

∎

Corollary

$\mathscr{L}\{\delta(t-0)\} = 1$.

Solving differential equations containing the Dirac delta function is similar with those without the Dirac delta function. The Dirac delta function comes out when one generates a differential equation with a function that is not differentiable at some point. The Dirac delta function in a differential equation doesn't contain a meaning itself, and something comes up only when one applies the Laplace transform. Since an exponential function comes out when you apply Laplace transform of the Dirac-delta function, the solution of the differential equation containing the Dirac-delta function contains unit-step functions.

Example 1

Solve $y'' + y = \delta(t-\pi)$, $y(0) = -2$, and $y'(0) = 0$.

Solution If you apply the Laplace transform for both sides, you get

$$s^2Y(s) + 2s + Y(s) - 0 = e^{-\pi s}.$$

Then, solving for $Y(s)$ gives you

$$(s^2+1)Y(s) = -2s + e^{-\pi s}$$

$$Y(s) = -2 \cdot \frac{s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

Using the inverse transform theorem, you get

$$y(t) = -2\cos t + \sin(t-\pi)\mathscr{U}(t-\pi) = \begin{cases} -2\cos t & t < 2\pi \\ -2\cos t - \sin t & t \geq \pi. \end{cases}$$

5.8 Systems of Differential Equations

Laplace transform can also be applied when solving systems of differential equations, mostly linear systems. After applying the Laplace transform, one can solve the system of algebraic equations, and then apply the inverse theorem to get the solution of the system.

Example 1

Solve

$$x' + y = \cos 2t$$

$$-x + y' = \sin 2t.$$

when $x(0) = 0$ and $y(0) = 0$.

Solution Applying Laplace transform for both equations, you obtain the system of equations

$$sX(s) - 0 + Y(s) = \frac{s}{s^2+4}$$
$$-X(s) + sY(s) - 0 = \frac{2}{s^2+4}$$

which is the same as

$$sX(s) + Y(s) = \frac{s}{s^2+4}$$
$$-X(s) + sY(s) = \frac{2}{s^2+4}.$$

Solving the system of algebraic equations of $X(s)$ and $Y(s)$ yields

$$X(s) = \frac{s^2-2}{(s^2+1)(s^2+4)} = -\frac{1}{s^2+1} + \frac{2}{s^2+4}$$
$$Y(s) = \frac{3s}{(s^2+1)(s^2+4)} = \frac{s}{s^2+1} - \frac{s}{s^2+4}.$$

Therefore, the solution is

$$x(t) = -\sin t + \sin 2t$$
$$y(t) = \cos t - \cos 2t.$$

Chapter 6

Systems of Differential Equations

In Section 2.3 we illustrated how to solve linear equations. Also, in Section 6.8, some linear systems were introduced. In this chapter, we focus on solving systems of differential equations, where there are n functions and n variables. We especially focus on first-order linear systems.

6.1 Theory of Linear Systems

First, what even is a first-order linear system? The term *first-order* and *linear* is the same as those that we defined earlier in Section 1.1.

Definition 6.1.1: First-order System

A **first-order system** is a set of first-order differential equations

$$\begin{aligned}\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \ldots, y_n)\\ \frac{dy_2}{dt} &= f_2(t, y_1, y_2, \ldots, y_n)\\ &\vdots\\ \frac{dy_n}{dt} &= f_n(t, y_1, y_2, \ldots, y_n).\end{aligned}$$

Definition 6.1.2: First-order Linear System

A first-order system is linear if it can be expressed in the form

$$\begin{aligned}\frac{dy_1}{dt} &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + g_1(t)\\ \frac{dy_1}{dt} &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + g_2(t)\\ &\vdots\\ \frac{dy_1}{dt} &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + g_1(t).\end{aligned}$$

The system above can be expressed in matrix form

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y} + \mathbf{G},$$

where

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \text{ and } \mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}.$$

Differentiation is defined entrywise.

Initial-Value Problems

An initial-value problem consists of the linear system

$$\mathbf{Y}' = \mathbf{AY} + \mathbf{G}$$

with an initial condition

$$\mathbf{Y}(x_0) = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} = \mathbf{Y}_0.$$

In section 1.2 and 3.1, we discussed whether there exists a unique solution to initial-value problems. There also exist a unique solution for initial-value problems in first-order linear systems.

Theorem 6.1.1: Existence and Uniqueness Theorem

If every entry of $\mathbf{A}$ and $\mathbf{G}$ is continuous on an interval containing x_0, then there exists a unique solution to the initial-value problem

$$\mathbf{Y}' = \mathbf{AY} + \mathbf{G}$$

$$\mathbf{Y}(t_0) = \mathbf{Y}_0$$

on the interval.

Homogeneous Systems

The superposition principle that we have discussed in section 3.1 also holds in linear systems. There were two kinds of superposition principles: for homogeneous equations and nonhomogeneous equations. Similar to how we defined earlier, a linear system is *homogeneous* if $\mathbf{G} = \mathbf{0}$. That is, if every entry of $\mathbf{G}$ is equal to zero.

Theorem 6.1.2: Superposition Principle

Let $\mathbf{Y}_1$, $\mathbf{Y}_2$, …, $\mathbf{Y}_n$ be solutions to the homogeneous linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}.$$

Then, for constants c_1, c_2, …, c_n,

$$c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n$$

is a solution to the homogeneous linear system.

Proof. Since $\mathbf{Y}_1$, $\mathbf{Y}_2$, …, $\mathbf{Y}_n$ are solutions to the homogeneous linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y},$$

we have

$$\begin{aligned}
\mathbf{Y}_1' &= \mathbf{A}\mathbf{Y}_1 \\
\mathbf{Y}_2' &= \mathbf{A}\mathbf{Y}_2 \\
&\vdots \\
\mathbf{Y}_n' &= \mathbf{A}\mathbf{Y}_n.
\end{aligned}$$

Substituting the linear combination gives

$$\begin{aligned}
(c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n)' &= c_1\mathbf{Y}_1' + c_2\mathbf{Y}_2' + \cdots + c_n\mathbf{Y}_n' \\
&= c_1\mathbf{A}\mathbf{Y}_1 + c_2\mathbf{A}\mathbf{Y}_2 + \cdots + c_n\mathbf{A}\mathbf{Y}_n \\
&= \mathbf{A}(c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n).
\end{aligned}$$

■

First-order homogeneous linear systems with n unknowns have n linear independent solutions.

Theorem 6.1.3: Existence of Linear Independent Solutions

The n solutions

$$\mathbf{Y}_1 = \begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{pmatrix}, \mathbf{Y}_2 = \begin{pmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{pmatrix}, \ldots, \mathbf{Y}_n = \begin{pmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{pmatrix}$$

of the homogeneous linear system $\mathbf{Y}' = \mathbf{AY}$ are linear independent if and only if

$$W(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \neq 0.$$

Definition 6.1.3: Fundamental Set of Solutions

If there are n linear independent solutions $\mathbf{Y}_1$, $\mathbf{Y}_2$, ..., $\mathbf{Y}_n$ to the homogeneous first-order linear system, then the set

$$\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n\}$$

is called the **fundamental set of solutions**.

Theorem 6.1.4: General Solution - Homogeneous System

If $\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n\}$ is the fundamental set of solutions to the homogeneous first-order linear system, then the **general solution** to the system is

$$\mathbf{Y}_c = c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n,$$

where c_1, c_2, ..., c_n are constants.

The general solution to the homogeneous system is called *complementary solution.*

Nonhomogeneous Systems

Recall the definition of a particular solution from section 3.1. Similarly, any solution to the first-order linear system

$$\mathbf{Y}' = \mathbf{AY} + \mathbf{G}$$

is called *particular solution* and denoted $\mathbf{Y}_p$.

Theorem 6.1.5: General Solution - Nonhomogeneous System

Let $\mathbf{Y}_p$ be any particular solution to the nonhomogeneous first-order linear system

$$\mathbf{Y}' = \mathbf{AY} + \mathbf{G}.$$

Then, the general solution to the system is

$$\begin{aligned}\mathbf{Y} &= \mathbf{Y}_c + \mathbf{Y}_p \\ &= c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n + \mathbf{Y}_p,\end{aligned}$$

where c_1, c_2, …, c_n are constants.

Conversion of a linear equation

A linear nth-order differential equation

$$y^{(n)} = a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y + f(t)$$

can be converted into a first-order linear system

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

by making the substitution

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{(n-1)}. \end{aligned}$$

Sometimes, solving a first-order linear system will be less complicated than solving a linear nth-order differential equation.

Example 1

Convert $y'' - 4y' + 3y = 0$ into a first-order linear system.

Solution Taking the substitution $y_1 = y$, $y_2 = y'$, the linear equation is converted to the system

$$\begin{aligned} y_1' &= \qquad\quad y_2 \\ y_2' &= -\,3y_1 + 4y_2, \end{aligned}$$

or

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

in matrix form.

6.2 Homogeneous Linear Systems

This section is focused on homogeneous first-order linear systems with constant coefficients, i.e. systems of the form

$$\mathbf{Y}' = \mathbf{AY}$$

where the entries in $\mathbf{A}$ are constants. We try a solution of the form

$$\mathbf{Y} = \mathbf{X}e^{\lambda t}.$$

We get $\mathbf{Y}' = \lambda \mathbf{X} e^{\lambda t} = \mathbf{AX}e^{\lambda t}$. Since $e^{\lambda t} \neq 0$, we get

$$\lambda \mathbf{X} = \mathbf{AX},$$

which is the eigenvalue problem. With $\mathbf{I}$ the identity matrix, rearranging terms gives

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}.$$

We want a nontrivial vector $\mathbf{X}$, so we must have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0.$$

The equation above is called the *characteristic equation*. In a system of n unknowns, the characteristic equation will be a nth-order polynomial equation with respect to λ. One can find the eigenvalue λ by finding the roots of the polynomial, and the corresponding eigenvector $\mathbf{X}$.

Case 1: Distinct Real Eigenvalues

If the characteristic equation possesses n distinct roots λ_1, λ_2, …, λ_n, then their corresponding eigenvectors $\mathbf{X}_1$, $\mathbf{X}_2$, …, $\mathbf{X}_n$ are linear independent. The proof for linear independence is stated below.

Lemma

For a matrix $\mathbf{A}$, if there exists two different eigenvalues λ to the eigenvalue problem

$$\lambda\mathbf{X} = \mathbf{A}\mathbf{X}$$

then the corresponding eigenvectors are linear independent.

Proof. Suppose there exists two eigenvalues λ_1 and λ_2, and their corresponding eigenvectors $\mathbf{X}_1$ and $\mathbf{X}_2$. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are linear dependent, then there exists a constant c such that

$$\mathbf{X}_2 = c\mathbf{X}_1.$$

Multiplying λ_2 to both sides gives

$$\lambda_2\mathbf{X}_2 = c\lambda_2\mathbf{X}_1.$$

Also, multiplying $\mathbf{A}$ to both sides gives

$$A\mathbf{X}_2 = cA\mathbf{X}_1$$

$$\lambda_2\mathbf{X}_2 = c\lambda_1\mathbf{X}_1.$$

Therefore, we have

$$\lambda_2\mathbf{X}_2 = c\lambda_2\mathbf{X}_1 = c\lambda_1\mathbf{X}_1,$$

which leads to $c(\lambda_1 - \lambda_2)\mathbf{X}_1 = \mathbf{0}$. However, since $c \neq 0$, $\lambda_1 \neq \lambda_2$, and $\mathbf{X}_1 \neq \mathbf{0}$, we have a contradiction, and the two eigenvectors are linear independent. ■

Therefore, the general solution is

$$\mathbf{Y} = c_1\mathbf{X}_1e^{\lambda_1 t} + c_2\mathbf{X}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{X}_ne^{\lambda_n t}.$$

Case 2: Repeated Eigenvalues
If an eigenvalue λ is repeated n times as a root. We call this *algebraic multiplicity.* If there exists n linear independent eigenvectors for λ, then we are done. However, in some cases, there would be less than n linear independent eigenvectors. In these cases, we should find new solutions to make n linear independent eigenvectors. We use the *generalized eigenvector* method.

Definition 6.2.1: Geometric Multiplicity

The **geometric multiplicity** of an eigenvalue is defined by the nullity of $\lambda I - A$.

The geometric multiplicity is the number of linear independent eigenvectors. Therefore, if the geometric multiplicity is not equal to the algebraic multiplicity, (note that the geometric multiplicity cannot exceed the algebraic multiplicity.) we need to find more eigenvectors that are linear independent. With knowing that

$\mathbf{X}_1 e^{\lambda t}$ is a solution to the system, we try a solution of the form

$$\mathbf{Y} = \mathbf{X}_1 t e^{\lambda t} + \mathbf{X}_2 e^{\lambda t}.$$

We have

$$\begin{aligned} \mathbf{Y}' = \mathbf{X}_1 e^{\lambda t} + &\lambda \mathbf{X}_1 t e^{\lambda t} + \lambda \mathbf{X}_2 e^{\lambda t} \text{ and} \\ \mathbf{A}\mathbf{Y} = \quad & \mathbf{A}\mathbf{X}_1 t e^{\lambda t} + \mathbf{A}\mathbf{X}_2 e^{\lambda t} \\ = \quad & \lambda \mathbf{X}_1 t e^{\lambda t} + \mathbf{A}\mathbf{X}_2 e^{\lambda t}. \end{aligned}$$

Therefore,

$$\mathbf{X}_1 e^{\lambda t} + \lambda \mathbf{X}_2 e^{\lambda t} = \mathbf{A}\mathbf{X}_2 e^{\lambda t}$$

and

$$\mathbf{X}_1 + \lambda \mathbf{X}_2 = \mathbf{A}\mathbf{X}_2.$$

$\mathbf{X}_2$ should satisfy the relation

$$(A - \lambda I)\mathbf{X}_2 = \mathbf{X}_1.$$

We call $\mathbf{X}_2$ the generalized eigenvector of A corresponding to λ. Therefore, for $\mathbf{X}_2$ such that $(A - \lambda I)\mathbf{X}_2 = \mathbf{X}_1$,

$$\mathbf{Y} = \mathbf{X}_1 t e^{\lambda t} + \mathbf{X}_2 e^{\lambda t}$$

is a solution to the linear system if the characteristic equation has repeated root λ. Then, $\mathbf{X}_1$ and $\mathbf{X}_2$ are linear independent.

Lemma

If $\mathbf{X}_1$ is the eigenvector corresponding to the eigenvalue λ and $\mathbf{X}_2$ is the generalized eigenvector satisfying

$$(A - \lambda I)\mathbf{X}_2 = \mathbf{X}_1,$$

then $\mathbf{X}_1$ and $\mathbf{X}_2$ are linear independent.

Proof. Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are linear dependent. That is, there exists constants c_1 and c_2, that are not all zero, satisfying

$$c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 = \mathbf{0}.$$

Multiplying $\lambda \mathbf{I} - \mathbf{A}$, we get

$$\begin{aligned} c_1(\lambda \mathbf{I} - \mathbf{A})\mathbf{X}_1 + c_2(\lambda \mathbf{I} - \mathbf{A})\mathbf{X}_2 &= c_1 \mathbf{0} + c_2 \mathbf{X}_1 \\ &= c_2 \mathbf{X}_1 = \mathbf{0}. \end{aligned}$$

Since $c_2\mathbf{X}_1 = \mathbf{0}$ but $\mathbf{X}_1 \neq \mathbf{0}$, $c_2 = 0$. Then, $c_1\mathbf{X}_1 = \mathbf{0}$ and we also have $c_1 = 0$. However, since c_1 and c_2 cannot be all zero, we have a contradiction. Therefore, $\mathbf{X}_1$ and $\mathbf{X}_2$ are linear independent. ∎

We now have obtained two linear independent solutions. We can repeat this process and get m linear independent solutions:

$$
\begin{aligned}
\mathbf{Y}_1 &= \mathbf{X}_1 e^{\lambda t} \\
\mathbf{Y}_2 &= \mathbf{X}_1 t e^{\lambda t} + \mathbf{X}_2 e^{\lambda t} \\
\mathbf{Y}_3 &= \mathbf{X}_1 \frac{t^2}{2} e^{\lambda t} + \mathbf{X}_2 t e^{\lambda t} + \mathbf{X}_3 e^{\lambda t} \\
&\vdots \\
\mathbf{Y}_m &= \mathbf{X}_1 \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \mathbf{X}_2 \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \cdots + \mathbf{X}_m e^{\lambda t}
\end{aligned}
$$

where

$\mathbf{X}_1$ is any eigenvector of λ,

$\mathbf{X}_2$ is generalized eigenvector such that $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X}_2 = \mathbf{X}_1$,

$\vdots$

$\mathbf{X}_m$ is generalized eigenvector such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X}_m = \mathbf{X}_{m-1}$.

Finally, the general solution contains

$$c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_m\mathbf{Y}_m.$$

Case 3: Complex Conjugate Eigenvalues
If the characteristic equation has complex conjugate roots $\alpha \pm i\beta$, we have two linear independent solutions. Denote these conjugate roots λ and $\bar{\lambda}$.

Lemma

If the characteristic equation has complex roots λ and $\bar{\lambda}$, then

$$\mathbf{X}e^{\lambda t} \text{ and } \overline{\mathbf{X}}e^{\bar{\lambda} t}$$

are both solutions to the homogeneous linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}.$$

Proof. It is evident that $\mathbf{X}e^{\lambda t}$ is a solution. Taking complex conjugation to the whole system

$$\mathbf{Y}' = \mathbf{AY},$$

we have

$$\begin{aligned}\overline{\mathbf{Y}}' &= \overline{\mathbf{AY}} \\ &= \mathbf{A}\overline{\mathbf{Y}}\end{aligned}$$

because A only contains real entries. Therefore,

$$\overline{\mathbf{Y}} = \overline{\mathbf{X}}e^{\bar{\lambda}t}$$

is also a solution. ■

Therefore, the two solutions are

$$\begin{aligned}\mathbf{X}e^{\lambda t} &= \mathbf{X}_1 e^{(\alpha+i\beta)t} = \mathbf{X}e^{\alpha t}(\cos\beta t + i\sin\beta t) \text{ and} \\ \overline{\mathbf{X}}e^{\bar{\lambda}t} &= \overline{\mathbf{X}}e^{(\alpha-i\beta)t} = \overline{\mathbf{X}}e^{\alpha t}(\cos\beta t - i\sin\beta t).\end{aligned}$$

Since these two solutions are linear independent, the linear combinations

$$\begin{aligned}\mathbf{Y}_1 &= \frac{1}{2}(\mathbf{X}e^{\lambda t} + \overline{\mathbf{X}}e^{\bar{\lambda}t}) \\ &= \frac{1}{2}(\mathbf{X} + \overline{\mathbf{X}})e^{\alpha t}\cos\beta t - \frac{i}{2}(-\mathbf{X} + \overline{\mathbf{X}})e^{\alpha t}\sin\beta t \text{ and} \\ \mathbf{Y}_2 &= \frac{i}{2}(\mathbf{X}e^{\lambda t} - \overline{\mathbf{X}}e^{\bar{\lambda}t}) \\ &= \frac{i}{2}(-\mathbf{X} + \overline{\mathbf{X}})e^{\alpha t}\cos\beta t + \frac{1}{2}(\mathbf{X} + \overline{\mathbf{X}})e^{\alpha t}\sin\beta t\end{aligned}$$

are also solutions to the linear system. If we let

$$\begin{aligned}\mathbf{B}_1 &= \frac{1}{2}(\mathbf{X} + \overline{\mathbf{X}}) = \Re(\mathbf{X}) \text{ and} \\ \mathbf{B}_2 &= \frac{i}{2}(-\mathbf{X} + \overline{\mathbf{X}})\ \Im(\mathbf{X}),\end{aligned}$$

then the two solutions to the system are

$$\begin{aligned}\mathbf{Y}_1 &= e^{\alpha t}(\mathbf{B}_1\cos\beta t - \mathbf{B}_2\sin\beta t) \text{ and} \\ \mathbf{Y}_2 &= e^{\alpha t}(\mathbf{B}_2\cos\beta t + \mathbf{B}_1\sin\beta t).\end{aligned}$$

Example 1

Solve $\mathbf{Y}' = \begin{pmatrix} -1 & 4 \\ 7 & 2 \end{pmatrix}\mathbf{Y}$.

Solution The characteristic equation is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda+1 & 4 \\ 7 & \lambda-2 \end{vmatrix} = 0,$$

which is $\lambda^2 - \lambda - 30 = 0$. Solving for λ, we have two distinct eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -5$. To find the corresponding eigenvectors, we solve the linear system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}.$$

For $\lambda_1 = 6$, we have

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X}_1 = \begin{pmatrix} -7 & 4 \\ 7 & -4 \end{pmatrix} \mathbf{X}_1 = \mathbf{0},$$

and hence

$$\mathbf{X}_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

For $\lambda_2 = -5$, we have

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X}_2 = \begin{pmatrix} 4 & 4 \\ 7 & 7 \end{pmatrix} \mathbf{X}_2 = \mathbf{0},$$

and hence

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the general solution to the system is

$$\begin{aligned} \mathbf{Y} &= c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}. \end{aligned}$$

Example 2

Solve $\mathbf{Y}' = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \mathbf{Y}$

Solution The characteristic equation is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda-3 & -1 \\ 1 & \lambda-5 \end{vmatrix} = 0,$$

which is $\lambda^2 - 8\lambda + 16 = 0$. Solving for λ, we have $\lambda = 4$ with algebraic multiplicity 2. To find the corresponding eigenvector, we solve the linear system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X}_1 = \mathbf{0}.$$

For $\lambda = 4$, we have

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X}_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X}_1 = \mathbf{0},$$

and hence

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

To find a second solution that is linear independent to the first, we let

$$\mathbf{Y}_2 = \mathbf{X}_1 te^{4t} + \mathbf{X}_2 e^{4t}.$$

Since $\mathbf{X}_2$ is the generalized eigenvector of $\mathbf{A}$, we have a linear system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X}_2 = \mathbf{X}_1.$$

Solving the linear system gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since we need any vector $\mathbf{X}_2$, we set $x_{12} = 0$ and $x_{22} = -1$. We get

$$mathbf X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and therefore the general solution is

$$\begin{aligned} \mathbf{Y} &= c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t} \right). \end{aligned}$$

Example 3

Solve $\mathbf{Y}' = \begin{pmatrix} 4 & -5 \\ 5 & 4 \end{pmatrix} \mathbf{Y}$.

Solution The characteristic equation is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 4 & -5 \\ 5 & \lambda - 4 \end{vmatrix} = 0,$$

which is $\lambda^2 - 8\lambda + 41 = 0$. Solving for λ, we have $\lambda = 4 \pm 5i$, which are complex conjugate roots. To find the corresponding eigenvector, we solve the linear system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}.$$

For $\lambda = 4 + 5i$, we have

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{X} = \begin{pmatrix} 5i & -5 \\ 5 & -5i \end{pmatrix} \mathbf{X} = \mathbf{0},$$

and hence

$$\mathbf{X} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Then,

$$\mathbf{B}_1 = \Re(\mathbf{X}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and}$$

$$\mathbf{B}_2 = \Im(\mathbf{X}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, the two solutions are

$$\begin{aligned}
\mathbf{Y}_1 &= e^{4t}(\mathbf{B}_1 \cos 5t - \mathbf{B}_2 \sin 5t) \\
&= e^{4t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 5t \right) \text{ and} \\
\mathbf{Y}_2 &= e^{4t}(\mathbf{B}_2 \cos 5t + \mathbf{B}_1 \sin 5t) \\
&= e^{4t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 5t \right),
\end{aligned}$$

and the general solution is

$$\begin{aligned}
\mathbf{Y} &= c_1 \mathbf{Y}_1 + c_2 \mathbf{Y}_2 \\
&= c_1 e^{4t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 5t \right) + c_2 e^{4t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 5t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 5t \right) \\
&= c_1 \begin{pmatrix} -\sin 5t \\ \cos 5t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \cos 5t \\ \sin 5t \end{pmatrix} e^{4t}.
\end{aligned}$$

6.3 Nonhomogeneous Linear Systems

For a nonhomogeneous first-order linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y} + \mathbf{G},$$

we are interested in the particular solution $\mathbf{Y}_p$ so that we can conclude that the general solution is

$$\mathbf{Y} = \mathbf{Y}_c + \mathbf{Y}_p$$

where $\mathbf{Y}_c$ is the solution to the homogeneous system

$$\mathbf{Y}' = \mathbf{AY}.$$

Undetermined Coefficient Method

Recall from section 3.5 that we simply guessed the solution to the linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

where $f(x)$ is either a polynomial, exponential function, sine or cosine function, and finite sums or products of these functions. The method goes the same with linear systems. Consider a nonhomogeneous linear system

$$\mathbf{Y}' = \mathbf{AY} + \mathbf{G}$$

where the entries of $\mathbf{G}(t)$ consist of polynomials, exponential functions, sine or cosine functions, and finite sums or products of these functions. Then one can guess the form of the particular solution can compare the coefficients to get the answer.

Example 1

Solve $\mathbf{Y}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 2t \\ 3t+2 \end{pmatrix}$.

Solution Since

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda - 3 \end{vmatrix},$$

the characteristic equation is $(\lambda - 4)(\lambda - 3) - 6 = \lambda^2 - 7\lambda + 6 = 0$. This gives two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 6$, and their corresponding eigenvectors are

$$\mathbf{X}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \text{ and } \mathbf{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the complementary solution is

$$\mathbf{Y}_c = c_1 \mathbf{X}_1 e^{\lambda_1 t} + c_2 \mathbf{X}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}.$$

For the particular solution, since

$$\mathbf{G}(t) = \begin{pmatrix} 2t \\ 3t+2 \end{pmatrix} = t\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

we guess the particular solution as

$$\mathbf{Y}_p = t\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}.$$

Substituting $\mathbf{Y}_p$ into the equation gives

$$\begin{aligned}\begin{pmatrix} a \\ b \end{pmatrix} &= t\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} 2t \\ 3t+2 \end{pmatrix} \\ &= t\begin{pmatrix} 4a+2b+2 \\ 3a+3b+3 \end{pmatrix} + \begin{pmatrix} 4c+2d \\ 3c+3d+2 \end{pmatrix}.\end{aligned}$$

Comparing the entries, we get

$$\begin{aligned} 4a+2b+2 &= 0 \qquad & 3a+3b+3 &= 0 \\ 4c+2d &= a \qquad & 3c+3d+2 &= b. \end{aligned}$$

Therefore, $a = 0$, $b = -1$, $c = 1$, $d = -2$, and the particular solution is

$$\mathbf{Y}_p = \begin{pmatrix} 1 \\ -t-2 \end{pmatrix}.$$

Finally, the general solution is

$$\mathbf{Y} = c_1\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t} + \begin{pmatrix} 1 \\ -t-2 \end{pmatrix}.$$

Even though the undetermined coefficient method seems useful, it is not as straightforward as the undetermined coefficient method in section 3.5. For example, if there are repeated eigenvalues, the form of $\mathbf{Y}_p$ may be inconsistent. We introduce a better method, called the variation of parameter method as introduced in section 3.6.

Variation of Parameter Method

Recall that if

$$\mathbf{Y}_1 = \begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{pmatrix}, \quad \mathbf{Y}_2 = \begin{pmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{pmatrix}, \quad \cdots \mathbf{Y}_n = \begin{pmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{pmatrix}$$

are n solutions to the homogeneous linear system

$$\mathbf{Y} = \mathbf{A}\mathbf{Y},$$

then

$$c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n$$

is also a solution. Notice that the general solution can be written as a product of two matrices

$$c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + \cdots + c_n\mathbf{Y}_n = \begin{pmatrix} c_1y_{11} + c_2y_{12} + \cdots + c_ny_{1n} \\ c_1y_{21} + c_2y_{22} + \cdots + c_ny_{2n} \\ \vdots \\ c_1y_{n1} + c_2y_{n2} + \cdots + c_ny_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{\Psi}\mathbf{C}.$$

Here, Ψ is called the *fundamental matrix.*

Definition 6.3.1: Fundamental Matrix

The **fundamental matrix** of a linear system is defined by

$$\Psi(t) = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}.$$

Then, since the fundamental matrix consists of n column vectors which are solutions to the linear system, the fundamental matrix satisfies

$$\mathbf{\Psi}' = \mathbf{A}\mathbf{\Psi}.$$

Lemma

$\det \mathbf{\Psi} \neq 0$, and there exists an inverse matrix of $\mathbf{\Psi}$.

Proof. Since every column of $\mathbf{\Psi}$ is a linearly independent solution to the equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X},$$

the determinant of $\mathbf{\Psi}$ is equivalent to the Wronskian of the column vectors, which cannot be zero.

$$\det \mathbf{\Psi} = W(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) \neq 0.$$

■

Assume that there exists a matrix

$$\mathbf{U} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

such that the particular solution to the nonhomogeneous linear system

$$\mathbf{Y} = \mathbf{A}\mathbf{Y} + \mathbf{G}.$$

can be expressed by

$$\mathbf{Y}_p = \mathbf{\Psi}\mathbf{U}.$$

Substituting $\mathbf{Y}_p$ into the system gives

$$\begin{aligned} \mathbf{Y}_p' &= \mathbf{\Psi}\mathbf{U}' + \mathbf{\Psi}'\mathbf{U} \\ &= \mathbf{\Psi}\mathbf{U}' + \mathbf{A}\mathbf{\Psi}\mathbf{U} \\ &= \mathbf{A}\mathbf{\Psi}\mathbf{U} + \mathbf{G}, \end{aligned}$$

and hence

$$\mathbf{\Psi}\mathbf{U}' = \mathbf{G}.$$

Since $\mathbf{\Psi}$ has an inverse, $\mathbf{U}$ can be found:

$$\begin{aligned} \mathbf{U}' &= \mathbf{\Psi}^{-1}\mathbf{G} \\ \mathbf{U} &= \int \mathbf{\Psi}^{-1}\mathbf{G}. \end{aligned}$$

Therefore, the particular solution is

$$\mathbf{Y}_p = \mathbf{\Psi} \int \mathbf{\Psi}^{-1}\mathbf{G},$$

and the general solution is

$$\mathbf{Y} = \mathbf{\Psi}\mathbf{C} + \mathbf{\Psi} \int \mathbf{\Psi}^{-1}\mathbf{G}$$

where integration is defined entrywise.

Example 2

Solve $\mathbf{Y}' = \begin{pmatrix} 5 & 1 \\ 2 & 6 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} -4t + 6 \\ 10t - 4 \end{pmatrix}$.

Solution Since

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 5 & -1 \\ -2 & \lambda - 6 \end{vmatrix},$$

the characteristic equation is $(\lambda - 5)(\lambda - 6) - 2 = \lambda^2 - 11\lambda + 28 = 0$. This gives two distinct eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 7$, and their corresponding eigenvectors are

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{X}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore, the complementary solution is

$$\mathbf{Y}_c = c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{7t}.$$

The fundamental matrix for the system is

$$\mathbf{\Psi} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix},$$

and its inverse $\mathbf{\Psi}^{-1}$ is

$$\mathbf{\Psi}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the particular solution is

$$\begin{aligned}
\mathbf{Y}_p &= \mathbf{\Psi} \int \mathbf{\Psi}^{-1}\mathbf{G} \\
&= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \int \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -4t + 6 \\ 10t - 4 \end{pmatrix} dt \\
&= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \int \begin{pmatrix} 6t + 2 \\ 24t - 14 \end{pmatrix} dt \\
&= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3t^2 + 2t \\ 12t^2 - 14t \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 15t^2 - 12t \\ 21t^2 - 30t \end{pmatrix} \\
&= \begin{pmatrix} 5t^2 - 4t \\ 7t^2 - 10t \end{pmatrix},
\end{aligned}$$

and the general solution to the system is

$$\begin{aligned}
\mathbf{Y} &= \mathbf{Y}_c + \mathbf{Y}_p \\
&= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{7t} + \begin{pmatrix} 5t^2 - 4t \\ 7t^2 - 10t \end{pmatrix}.
\end{aligned}$$

6.4 The Exponential Matrix

Recall that the first-order differential equation

$$y'(x) = cy(x)$$

has a solution $y(x) = e^{cx}$. We can approach similarly for linear systems. For a linear system

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

where $\mathbf{A}$ has constant entries, we use the *exponential matrix* method. Recall the Taylor series of e^x:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

The exponential matrix is defined similarly.

Definition 6.4.1: Exponential Matrix

For a $n \times n$ matrix with constant entries, the exponential matrix is defined by

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots.$$

Similarly, we define

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 + \cdots \\ &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots. \end{aligned}$$

Note that we can commute $\mathbf{A}$ and t since $\mathbf{A}$ is a matrix and t is a variable, so it's a scalar.

Computing Exponential Matrices

Example 1

If $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, then find $e^{\mathbf{A}t}$.

Solution The matrix powers of $\mathbf{A}$ are

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{A}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

$$\vdots.$$

Therefore,

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} at & 0 \\ 0 & bt \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} a^2t^2 & 0 \\ 0 & b^2t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} a^3t^3 & 0 \\ 0 & b^3t^3 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots & 0 \\ 0 & 1 + bt + \frac{1}{2!}b^2t^2 + \frac{1}{3!}b^3t^3 + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}. \end{aligned}$$

Theorem 6.4.1: Exponential Matrix for Diagonal Matrices

Let $\mathbf{A}$ be a diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Then, the exponential matrix for $\mathbf{A}$ is

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{a_{11}t} & 0 & \cdots & 0 \\ 0 & e^{a_{22}t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_{nn}t} \end{pmatrix}.$$

How do we compute exponential matrices when $\mathbf{A}$ is not diagonal? For those cases, we use the Laplace transform. We will soon prove that the exponential matrix $e^{\mathbf{A}t}$ is a solution to the initial-value problem $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$, $\mathbf{Y}(0) = \mathbf{I}$. If we let $\mathbf{X}(s) = \mathscr{L}\{\mathbf{Y}(t)\} = \mathscr{L}\{e^{\mathbf{A}t}\}$, then

$$\begin{aligned}\mathscr{L}\{\mathbf{Y}'\} &= \mathscr{L}\{\mathbf{A}\mathbf{Y}\}\\ s\mathbf{X}(s) - \mathbf{Y}(0) &= \mathbf{A}\mathbf{X}(s).\end{aligned}$$

Solving for $\mathbf{X}(s)$, we have

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}.$$

Therefore, we can compute the exponential matrix

$$e^{\mathbf{A}t} = \mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}.$$

Example 2

Compute $e^{\mathbf{A}t}$ where $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Solution We have

$$\begin{aligned}
e^{\mathbf{A}t} &= \mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}\\
&= \mathscr{L}^{-1}\left\{\begin{pmatrix} s-2 & 1 \\ 1 & s-2 \end{pmatrix}^{-1}\right\}\\
&= \mathscr{L}^{-1}\left\{\begin{pmatrix} \dfrac{s-2}{(s-1)(s-3)} & \dfrac{-1}{(s-1)(s-3)} \\ \dfrac{-1}{(s-1)(s-3)} & \dfrac{s-2}{(s-1)(s-3)} \end{pmatrix}\right\}\\
&= \frac{1}{2}\mathscr{L}^{-1}\left\{\begin{pmatrix} \dfrac{1}{s-1} + \dfrac{1}{s-3} & \dfrac{1}{s-1} - \dfrac{1}{s-3} \\ \dfrac{1}{s-1} - \dfrac{1}{s-3} & \dfrac{1}{s-1} + \dfrac{1}{s-3} \end{pmatrix}\right\}\\
&= \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & e^t - e^{3t} \\ e^t - e^{3t} & e^t + e^{3t} \end{pmatrix}.
\end{aligned}$$

Solving Differential Equations

Derivatives are done entrywise. Since t is a scalar, $\mathbf{A}$ does not have any effects when differentiating with respect to t. This means that, for example,

$$\frac{d}{dt}\mathbf{A}^n t^n = n\mathbf{A}^n t^{n-1}.$$

Lemma

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}.$$

Proof.

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{A}t} &= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\right)\\ &= \mathbf{0} + \mathbf{A} + \frac{2}{2!}\mathbf{A}^2t + \frac{3}{3!}\mathbf{A}^3t^2 + \cdots\\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\right) = \mathbf{A}e^{\mathbf{A}t}.\end{aligned}$$ ■

Theorem 6.4.2: The Fundamental Matrix

For a homogeneous linear first-order system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}$$

where $\mathbf{A}$ consists of constant entries, $e^{\mathbf{A}t}$ is a fundamental matrix.

Proof. If is evident that $e^{\mathbf{A}t}$ is a solution to the system $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$. Since $\mathbf{Y}(0) = e^{\mathbf{A}0} = \mathbf{I}$, we have $\det e^{\mathbf{A}0} \neq 0$. Since the determinant is equal to the Wronskian of n column vectors, the exponential matrix $e^{\mathbf{A}0}$ is a fundamental matrix. ■

The solution to the nonhomogeneous linear first-order system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y} + \mathbf{G}$$

can be expressed as

$$\mathbf{Y} = \mathbf{\Psi}\mathbf{C} + \mathbf{\Psi}\int \mathbf{\Psi}^{-1}\mathbf{G}.$$

Example 3

Solve $\mathbf{Y}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

Solution We have

$$\mathbf{Y} = \mathbf{\Psi C} + \mathbf{\Psi}\int \mathbf{\Psi^{-1} G}$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\int \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\int \begin{pmatrix} -3e^{-3t} \\ 2e^{-2t} \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} e^{-3t} \\ -e^{-2t} \end{pmatrix}$$

$$= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

6.5 Autonomous Systems

Until now, we have looked at how to solve some linear systems. From now on, we look at the system's stability, or how it behaves when t tends to infinity. We focus on the analysis of *autonomous systems*.

Autonomous Systems

Definition 6.5.1: Autonomous Systems

A first-order system

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(t, y_1, y_2, \dots, y_n) \\ \frac{dy_2}{dt} &= f_2(t, y_1, y_2, \dots, y_n) \\ \vdots \quad & \qquad \vdots \\ \frac{dy_n}{dt} &= f_1(t, y_1, y_2, \dots, y_n) \end{aligned}$$

is **autonomous** if the functions f_1, f_2, …, f_n are independent on t. That is, if the system can be represented as

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(y_1, y_2, \dots, y_n) \\ \frac{dy_2}{dt} &= f_2(y_1, y_2, \dots, y_n) \\ \vdots \quad & \qquad \vdots \\ \frac{dy_n}{dt} &= f_n(y_1, y_2, \dots, y_n). \end{aligned}$$

If we set

$$\mathbf{X}(t) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n, \end{pmatrix} \text{ and } \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$

the autonomous system can be represented as

$$\mathbf{X}(t)' = \mathbf{g}(\mathbf{X}(t)).$$

The right-hand side is a vector field. If we set $\mathbf{X}(t)$ as the position vector in a n-dimensional space, then $\mathbf{X}(t)'$ is the velocity vector. The initial condition is where the particle starts, and is denoted by

$$\mathbf{X}(0) \text{ or } \mathbf{X}_0.$$

This initial-value problem has a unique solution locally.

Theorem 6.5.1: Existence and Uniqueness Theorem

Consider a first-order autonomous system

$$\begin{aligned}
\frac{dy_1}{dt} &= f_1(y_1,y_2,\ldots,y_n) \\
\frac{dy_2}{dt} &= f_2(y_1,y_2,\ldots,y_n) \\
\vdots \quad & \quad \vdots \\
\frac{dy_n}{dt} &= f_1(y_1,y_2,\ldots,y_n).
\end{aligned}$$

with initial condition $\mathbf{X}(t_0) = \mathbf{X}_{t_0}$. If f_1, f_2, ..., f_n are continuous and have continuous first partial derivatives, then there is a unique solution to the system on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

Even though the uniqueness is not satisfied globally, we assume that there is a globally unique solution.

Three Types of Solutions

Given a first-order autonomous system and initial condition $\mathbf{X}(0)$, there are three types of solutions.

Type 1: Critical Point

Critical point solutions are constant solutions or a stationary point for all t. If the initial condition $\mathbf{X}_0$ is a critical point, then the particle stays there. Since the particle is stationary, $\mathbf{X}(t)' = \mathbf{g}(\mathbf{X}(t)) = 0$, and a critical point is a solution to algebraic equations

$$\begin{aligned}
g_1(\mathbf{X}) &= 0 \\
g_2(\mathbf{X}) &= 0 \\
&\vdots \\
g_n(\mathbf{X}) &= 0.
\end{aligned}$$

Type 2: Arc
Generally, a solution $\mathbf{X}(t)$ is an arc. Note that this curve is *simple*: it cannot cross itself. If the curve has an intersection $\mathbf{P}$ with itself, then the solution to the autonomous system with initial condition $\mathbf{X}(t_p) = \mathbf{P}$ will not be unique.

Type 3: Cycle
If a solution is periodic, that is, if there exists a real number $T > 0$ such that $\mathbf{X}(t+T) = \mathbf{X}(t)$, then the solution will form a cycle and return to $\mathbf{X}_0$.

Example 1

Classify the types of the solutions to the first-order autonomous systems below.

$$\begin{aligned} 1. \qquad x' &= x - y \\ y' &= x^2 + y^2 - 8 \\ \mathbf{X}(0) &= (2,2) \\ 2. \qquad x' &= 5x + y \\ y' &= 2x + 6y \\ \mathbf{X}(0) &= (2,1) \\ 3. \qquad x' &= x + y \\ y' &= -2x - y \\ \mathbf{X}(0) &= (-5,4) \end{aligned}$$

Solution 1. Substituting the initial condition $\mathbf{X}(0) = (2,2)$ into the system gives

$$\begin{aligned} x' &= 2 - 2 = 0 \\ y' &= 2^2 + 2^2 - 8 = 0, \end{aligned}$$

therefore $(2,2)$ is a critical point.

2. Substituting the initial condition, one see that the initial condition $(2,1)$ is not a critical point. However, since the system is linear, we can actually solve the system and look for the type of solution. Solving the system gives

$$\begin{aligned} x &= c_1 e^{4t} + c_2 e^{7t} \\ y &= -c_1 e^{4t} + 2c_2 e^{7t}. \end{aligned}$$

Applying the initial condition $\mathbf{X}(0) = (2,1)$, we have $c_1 = 1$, $c_2 = 1$, and the solution to the system is

$$x = e^{4t} + e^{7t}$$
$$y = -e^{4t} + 2e^{7t}.$$

Since the solution is not periodic, it is an arc.

3. The system is also linear, so solving the system gives

$$x = c_1(\sin t - \cos t) + c_2(-\cos t - \sin t)$$
$$y = c_1 \cos t + c_2 \sin t.$$

Applying initial condition, we have $c_1 = 2$ and $c_2 = 3$. Therefore, the solution to the system is

$$x = -5\cos t - \sin t$$
$$y = 2\cos t + 3\sin t,$$

which is a cycle with period 2π since $\cos t$ and $\sin t$ is periodic with period 2π.

6.6 Stability of Linear Systems

Consider a linear autonomous system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy,$$

or

$$\mathbf{X}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{X}(t) = \mathbf{A}\mathbf{X}(t)$$

in matrix form. We are interested how $\mathbf{X}(t)$ behaves as t goes to infinity, or $\lim_{t\to\infty} \mathbf{X}(t)$. Since $(x, y) = (0, 0)$ is a solution to the linear system

$$\frac{dx}{dt} = ax + by = 0$$
$$\frac{dy}{dt} = cx + dy = 0,$$

we use $(0, 0)$ as the critical point to analyze. There are three cases. The particle may return to a critical point, remain close to a critical point if the solution is periodic, or move away from the critical point. For the first two cases, we call the critical point **locally stable**; for the third case, we call the critical point **unstable**. The stability depends on the eigenvalues λ.

The characteristic equation is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0.$$

Rearranging terms, we have

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = a + d = \text{tr}(\mathbf{A}) \text{ and } q = ad - bc = \det(\mathbf{A}).$$

By the quadratic formula, we get that the two eigenvalues are

$$\lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

Case 1: Distinct Real Eigenvalues

If there are two distinct real eigenvalues, i.e. when $p^2 - 4q > 0$, then the general solution is

$$\mathbf{X}(t) = c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_2 t}.$$

Without loss of generality, let $\lambda_1 > \lambda_2$. Then, the solution can be expressed as

$$\begin{aligned}\mathbf{X}(t) &= c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_2 t} \\ &= e^{\lambda_1 t}(c_1\mathbf{X}_1 + c_2\mathbf{X}_2 e^{(\lambda_2 - \lambda_1)t}).\end{aligned}$$

Here, we again divide into three cases, depending on the signs of λ_1 and λ_2.

1. If λ_1 and λ_2 are both positive, i.e. if p and q are both positive, $\lim_{t\to\infty} \mathbf{X}(t)$ diverges, therefore the critical point is **unstable**.

2. If λ_1 and λ_2 are both negative, i.e. if $p < 0$ and $q > 0$, then $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$, and the critical point is **stable**.

3. If λ_1 is positive and λ_2 is negative, i.e. if $q < 0$, then $\lim_{t\to\infty} \mathbf{X}(t) = c_1\mathbf{X}_1 e^{\lambda_1 t}$ if $c_1 \neq 0$, but $\lim_{t\to\infty} \mathbf{X}(t) = 0$ if $c_1 = 0$, so the critical point is **saddle**.

Case 2: Repeated Eigenvalues

If there are repeated eigenvalues, i.e. when $p^2 - 4q = 0$, then we divide into two cases where if there are two linear independent eigenvectors or only one linear independent eigenvector.

1. If there are two linear eigenvectors, then the general solution is

   $$\mathbf{X}(t) = c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_1 t}.$$

   If λ_1 is negative, i.e. if $p < 0$, then $\lim_{t\to\infty} = \mathbf{0}$, and the critical point is called a **degenerate stable node**.

If λ_1 is positive, i.e. if $p > 0$, then $\lim_{t\to\infty}$ diverges, and the critical point is called a **degenerate unstable node**.

2. If there is one linear independent eigenvector, then the general solution is

$$\mathbf{X}(t) = c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2(\mathbf{X}_1 t e^{\lambda_1 t} + \mathbf{X}_2 e^{\lambda_1 t}).$$

If λ_1 is negative, i.e. if $p < 0$, then $\lim_{t\to\infty} = \mathbf{0}$, and the critical point is called a **degenerate stable node**.
If λ_1 is positive, i.e. if $p > 0$, then $\lim_{t\to\infty}$ diverges, and the critical point is called a **degenerate unstable node**.

Therefore, we have that the critical point is a **degenerate stable node** if $p < 0$ and a **degenerate unstable node** if $p > 0$ independently of the number of linear independent eigenvectors.

Case 3: Complex Conjugate Eigenvalues
If there are complex conjugate eigenvalues $\alpha \pm i\beta$, i.e. when $p^2 - 4q < 0$, then the general solution is

$$\mathbf{X}(t) = c_1 e^{\alpha t}(\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t) + c_2 e^{\alpha t}(\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t).$$

We now divide into three cases.

1. If the roots are pure imaginary, i.e. if $\alpha = 0$, then

$$\mathbf{X}(t) = c_1(\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t) + c_2(\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t),$$

which is periodic. We then call the critical point **center**.

2. If the real parts are negative, i.e. if $\alpha < 0$ and $p < 0$, then $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{0}$, and the critical point is **stable spiral**.

3. If the real parts are positive, i.e. if $\alpha > 0$ and $p > 0$, then $\lim_{t\to\infty} \mathbf{X}(t)$ diverges, and the critical point is **unstable spiral**.

The cases above can be summarized geometrically. The figure below is called the **stability chart**, and it shows the summary of stability depending on p and q.

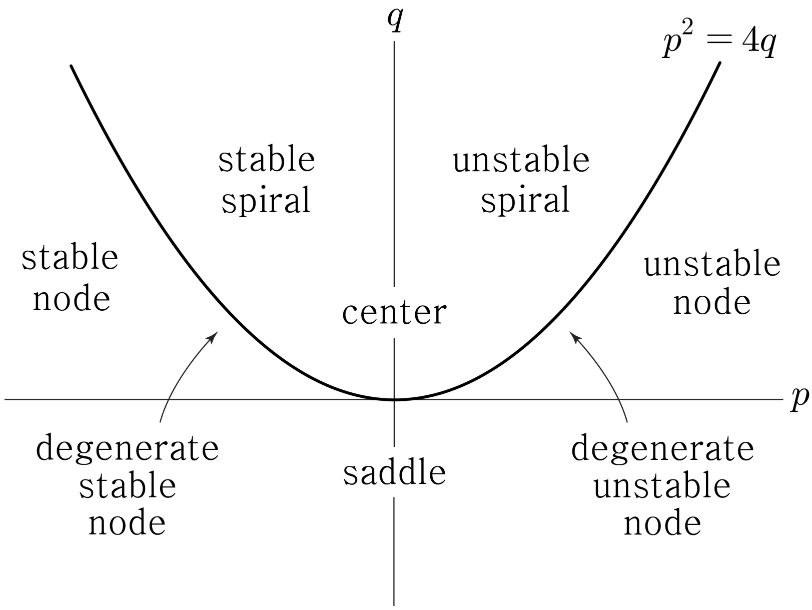


Figure 6.1: Stability Chart for Linear Systems

Example 1

Classify the critical point $(0, 0)$ of each linear system.

1. $\mathbf{X}'(t) = \begin{pmatrix} -1 & 4 \\ 7 & 2 \end{pmatrix} \mathbf{X}(t)$
2. $\mathbf{X}'(t) = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \mathbf{X}(t)$
3. $\mathbf{X}'(t) = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \mathbf{X}(t)$
4. $\mathbf{X}'(t) = \begin{pmatrix} 4 & -5 \\ 5 & 4 \end{pmatrix} \mathbf{X}(t)$

Solution 1. We have $p = 1$ and $q = -30$. Since $q < 0$, the critical point is saddle.

2. We have $p = 8$ and $q = 16$. Since $p^2 = 4q$ and $p > 0$, the critical point is degenerate unstable node.

3. We have $p = 0$ and $q = 9$. Since $p = 0$ and $q > 0$, the critical point is center.

4. We have $p = 8$ and $q = 41$. Since $p^2 < 4q$, $p > 0$, and $q > 0$, the critical point is unstable spiral.

6.7 Stability of Nonlinear Systems

Unlike autonomous linear systems where $(0,0)$ was always a critical point, autonomous nonlinear systems do not have a specific critical point that works for every system. There may be none, one, or many critical points. For example, in example 1 in section 6.4, any point on $y = x$ is a critical point for the first system. The problem with multiple critical points is this: when the initial condition $\mathbf{X}_0$ is not close enough to a critical point $\mathbf{X}_1$, then it may go towards another critical point $\mathbf{X}_2$ instead of $\mathbf{X}_1$. Therefore, we need to classify critical points. Critical points for autonomous nonlinear systems can be classified into *stable critical points* and *unstable critical points*.

Definition 6.7.1: Stable Critical Point

A critical point $\mathbf{X}_1$ of an autonomous system is a **stable critical point** if given any radius $\rho > 0$, there exists $r > 0$ such that if the initial condition satisfies $|\mathbf{X}_0 - \mathbf{X}_1| < r$, then $|\mathbf{X}(t) - \mathbf{X}_1| < \rho$ for all t.

Definition 6.7.2: Asymptotically Stable Critical Point

In the definition above, if $\lim_{t\to\infty} \mathbf{X}(t) = \mathbf{X}_1$ whenever $|\mathbf{X}_0 - \mathbf{X}_1| < r$, then the critical point $\mathbf{X}_1$ is called an **asymptotically stable critical point**.

Definition 6.7.3: Unstable Critical Point

A critical point $\mathbf{X}_1$ of an autonomous system is an **unstable critical point** if there exists some radius $\rho > 0$ such that for any $r > 0$, there exists an initial condition $\mathbf{X}_0$ such that $|\mathbf{X}_0 - \mathbf{X}_1| < r$ yet $|\mathbf{X}(t) - \mathbf{X}_1| \geq \rho$ for at least one $t > 0$.

Note that if a critical point is asymptotically stable, then it is stable.

Linearization

We start with a first-order equation $x' = g(x)$. Let x_1 be a critical point which we want to know whether it is stable or not. Since it is hard to solve a nonlinear equation $x' = g(x)$, we use the tangent line approximation

$$g(x) \approx g'(x_1)(x - x_1).$$

Then, the equation becomes

$$x' = g(x) \approx g'(x_1)(x - x_1).$$

Solving for x, we have

$$x = x_1 + e^{g'(x_1)t}.$$

We can now conclude that if $g'(x_1) < 0$, then x_1 is a stable critical point, and if $g'(x_1) > 0$, then x_1 is an unstable critical point.

Example 1

Determine whether the critical point π of a first-order differential equation

$$x' = \sin x$$

is stable or unstable.

Solution We have an approximation

$$x' = \sin x \approx -(x - \pi).$$

Therefore, solving for x gives

$$x = \pi + e^{-t}.$$

Since $-1 < 0$, the critical point π is stable.

Recall the formula for the tangent plane approximation of a function $z = f(x, y)$ near (x_1, y_1)

$$z \approx f(x_1, y_1) + f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1).$$

For a nonlinear system

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

where $\mathbf{X}_1 = (x_1, y_1)$ is a critical point satisfying $f(x_1, y_1) = g(x_1, y_1) = 0$, we have two linearization formulas

$$x' = f(x, y) \approx f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1)$$

$$y' = g(x, y) \approx g_x(x_1, y_1)(x - x_1) + g_y(x_1, y_1)(y - y_1).$$

Therefore, the system can be represented as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ g_x(x_1, y_1) & g_y(x_1, y_1) \end{pmatrix} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix},$$

which can be summarized as

$$\mathbf{X}' = \mathbf{A}(\mathbf{X} - \mathbf{X}_1)$$

where $\mathbf{A}$ is the Jacobian matrix

$$\mathbf{A} = \begin{pmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ g_x(x_1, y_1) & g_y(x_1, y_1) \end{pmatrix}$$

and is denoted by $\mathbf{A} = \mathbf{f}(\mathbf{X}_1)$. For the Jacobian matrix $\mathbf{f}(\mathbf{X}_1)$, if the real parts of both eigenvalues are negative, then the critical point is stable, and if there is an eigenvalue with a positive real part, then the critical point is unstable. This is because if there is a positive real part α, the solution will contain $e^{\alpha t}$, which diverges to infinity when $t \to \infty$.

Example 2

Classify the critical points as stable or unstable.

$$\frac{dx}{dt} = x^2 + y^2 - 20$$
$$\frac{dy}{dt} = x - y^2$$

Solution The critical points are points that satisfy

$$x^2 + y^2 - 20 = 0$$
$$x - y^2 = 0,$$

which is $(4, 2)$ and $(4, -2)$. For $(4, 2)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(4,2) & f_y(4,2) \\ g_x(4,2) & g_y(4,2) \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 1 & -4 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 - 4\lambda - 36 = 0,$$

which gives $\lambda = 2 \pm \sqrt{40}$. Since there is an eigenvalue with a positive real part, the critical point $(4, 2)$ is stable. For $(4, -2)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(4,-2) & f_y(4,-2) \\ g_x(4,-2) & g_y(4,-2) \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 - 12\lambda + 36 = 0,$$

which gives $\lambda = 6$. Since there is an eigenvalue with a positive real part, the critical point $(4, -2)$ is stable.

Not only classifying critical points as stable or unstable, we're able to classify types of critical points, as in section 6.5. The summarized stability chart is drawn below.

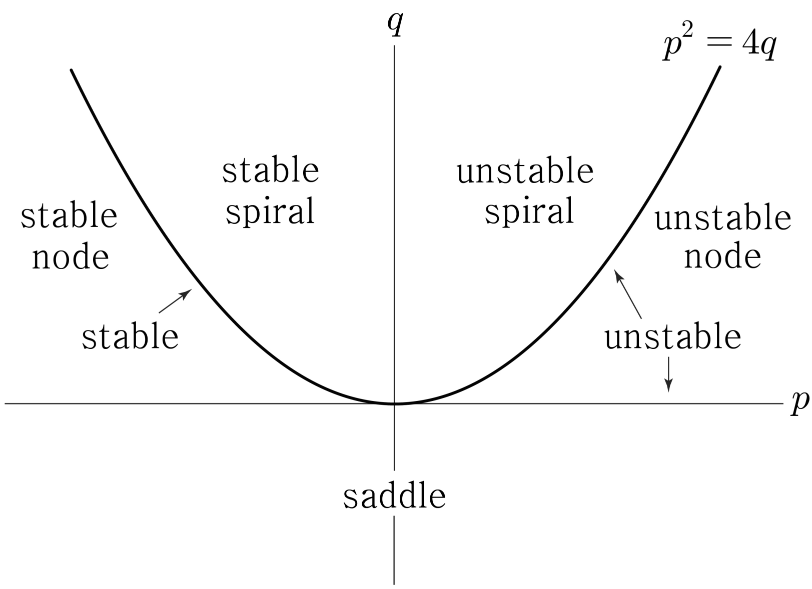


Figure 6.2: Stability Chart for Nonlinear Systems

Notice that center, degenerated stable node, and degenerated unstable node are not mentioned. This is because the formulas in the borderline $p^2 = 4q$ or $q = 0$ are obtained by tangent plane approximations, so the actual function may not be on the borderline. Therefore, we are unable to classify if the characteristic equation of the linearization satisfies $p^2 = 4q$ or $q = 0$, and we can only say if the critical point is stable or unstable. Also, if $p = 0$, we cannot even conclude that the critical point is stable or unstable. We should use another method. For the other five cases, the critical point can be categorized the same as what we have done for linear systems, and they also have the same geometrical properties.

Example 3

Classify the critical points as stable or unstable.

$$\frac{dx}{dt} = x^2 - y^2$$

$$\frac{dy}{dt} = x + 2y - 3$$

Solution The critical points are points that satisfy

$$x^2 - y^2 = 0$$

$$x + 2y - 3 = 0,$$

which is $(1, 1)$ and $(-3, 3)$. For $(1, 1)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(1,1) & f_y(1,1) \\ g_x(1,1) & g_y(1,1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 - 4\lambda + 6 = 0.$$

Since $p = 4$ and $q = 6$, we have $p > 0$, $q > 0$, and $p^2 < 4q$. Therefore, the critical point is unstable spiral. For $(-3, 3)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(-3,3) & f_y(-3,3) \\ g_x(-3,3) & g_y(-3,3) \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -6 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 + 4\lambda - 6 = 0.$$

Since $p = -4$ and $q = -6 < 0$, the critical point is saddle.

Example 4

Classify the critical points as stable or unstable.

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = 5x - 2y - 3$$

Solution The critical points are points that satisfy

$$x^2 - y = 0$$
$$5x - 2y - 3 = 0,$$

which is $(1, 1)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$. For $(1, 1)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(1,1) & f_y(1,1) \\ g_x(1,1) & g_y(1,1) \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 + 1 = 0.$$

Since $p = 0$, we cannot classify the type of the critical point. For $(3/2, 9/4)$, the Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x\left(\frac{3}{2}, \frac{9}{4}\right) & f_y\left(\frac{3}{2}, \frac{9}{4}\right) \\ g_x\left(\frac{3}{2}, \frac{9}{4}\right) & g_y\left(\frac{3}{2}, \frac{9}{4}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 - \lambda - 1 = 0.$$

Since $p = 1$ and $q = -1 < 0$, the critical point is saddle.

The Phase Plane Method

Then, how should we classify critical points when (p, q) is in the borderline? One approach can be done, which is called the phase plane method. Consider an autonomous nonlinear system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y).$$

The method is to actually solve the equation. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x, y)}{f(x, y)}.$$

If we are lucky enough, then we may be able to get a separable first-order equation and find the solution.

Example 5

Classify the critical point as stable or unstable.

$$\frac{dx}{dt} = y^2$$
$$\frac{dy}{dt} = x^2$$

Solution The critical points are points that satisfy

$$y^2 = 0$$
$$x^2 = 0,$$

which is $(0, 0)$. The Jacobian matrix is

$$\mathbf{f}(\mathbf{X}_1) = \begin{pmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues are roots to the equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2 = 0.$$

Since $p = 0$, we cannot classify the type of the critical point. We try to classify the critical point by using the phase plane method. Since

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt}\\ &= \frac{g(x,y)}{f(x,y)} = \frac{x^2}{y^2},\end{aligned}$$

we have a separable equation

$$y^2\,dy = x^2\,dx.$$

Solving the equation, we get

$$\begin{aligned}\int y^2\,dy &= \int x^2\,dx\\ \frac{1}{3}y^3 &= \frac{1}{3}x^3 + c\\ y &= \sqrt[3]{x^3 + c}.\end{aligned}$$

One can see that the particle moves away from the origin as t increases. Therefore, $(0,0)$ is unstable.

6.8 Limit Cycles and Periodic Solutions

For linear equations, the critical point was a center if the eigenvalues were pure imaginary. However, for nonlinear systems, we cannot conclude that a critical point is a center because there might be an error in the linearization. This section covers periodic solutions to nonlinear systems. The study of periodic solutions in nonlinear systems involves *limit cycles*. We let

$$\mathbf{V}(x,y) = \big(f(x,y), g(x,y)\big)$$

the velocity vector field of the autonomous system

$$\begin{aligned}x' &= f(x,y)\\ y' &= g(x,y).\end{aligned}$$

Definition 6.8.1: Limit Cycle

A **limit cycle** is a closed trajectory having the property that at least one other trajectory spirals into it, either as time approaches infinity or as time approaches negative infinity.

If there is a closed curve C, the nearby curves may also behave like C, just not closed. They may spiral towards C, spiral away from C, or both. If there is at least one curve such spirals, then C is called a limit cycle. Limit cycles can be classified into *stable*, *unstable*, or *semi-stable*.

Definition 6.8.2: Stable, Unstable, and Semi-Stable Limit Cycles

Let C be a limit cycle. If the trajectories nearby C spiral towards C, then C is a **stable limit cycle**. If the trajectories nearby C spiral away from C, then C is an **unstable limit cycle**. If the trajectories both spiral towards and away, then C is a **semi-stable limit cycle**.

Note that stable limit cycles consist of trajectories that spiral into C when time approaches infinity, and unstable limit cycles consist of trajectories into C when time approaches negative infinity.

Existence of Limit Cycles

When will limit cycles exist? The Poincare-Bendixson theorem illustrates a condition for a limit cycle to exist.

Definition 6.8.3: Invariant Region

For an autonomous system

$$x' = f(x, y)$$

$$y' = g(x, y),$$

the region R is called an **invariant region** if the solution $\mathbf{X}(t)$ for the system with the initial condition $\mathbf{X}(0) = \mathbf{X}_0$ stays inside R whenever $\mathbf{X}_0$ is in R.

For invariant regions, the velocity vector $\mathbf{V}(x, y) = \big(f(x, y), g(x, y)\big)$ always points toward the interior of R, so that the particle always stay inside R.

Theorem 6.8.1: Poincare-Bendixson Theorem

Let R be an invariant region bounded by two curves C_1 and C_2. If R contains no critical points, then the autonomous system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

has a periodic solution in R.

The proof is omitted since it requires analysis. The idea is that the particle can never leave R since the velocity vectors point towards the interior of R. Therefore, the particle should follow a curve or approach a critical point as $t \to \infty$. However, there aren't any critical points in R, so the particle should follow a closed curve. The criterion for invariant regions is stated in the theorem below.

Theorem 6.8.2: Criterion for Invariant Regions

Let R be a bounded region. For each point (x, y) on the boundary of R, let $\mathbf{V}(x, y)$ be the velocity vector at (x, y) and $\mathbf{N}(x, y)$ be the normal vector at (x, y). For any point (x_1, y_1) on the boundary of R, if $\mathbf{V}(x_1, y_1) \cdot \mathbf{N}(x_1, y_1) \geq 0$, then R is an invariant region.

Proof. Given a region R, let $\theta(x, y)$ be the angle between $\mathbf{V}(x, y)$ and $\mathbf{N}(x, y)$. Then, for any point (x_1, y_1) on the boundary of R, we have

$$\mathbf{V}(x_1, y_1) \cdot \mathbf{N}(x_1, y_1) = |\mathbf{V}(x_1, y_1)||\mathbf{N}(x_1, y_1)| \cos\theta(x_1, y_1) \geq 0$$

and $\cos\theta(x_1, y_1) \geq 0$ since norms are positive. Therefore, $0° \leq \theta(x_1, y_1) \leq 90°$ for every point (x_1, y_1) inside R. This makes the particle impossible to leave R and hence R is an invariant region. ■

Example 1

Show that the system

$$\begin{aligned} x' &= -x + y + xy \\ y' &= x^2 + y \end{aligned}$$

has a periodic solution.

Solution We claim that the region R bounded with $1/4 \leq x^2 + y^2 \leq 4$ is an invariant region. Since the normal vector is $\mathbf{N}(x, y) = (-2x, 2y)$, we have

$$\begin{aligned} \mathbf{V}(x, y) \cdot \mathbf{N}(x, y) &= -2x(-x + y + xy) + 2y(x^2 + y) \\ &= 2x^2 - 2xy + 2y^2 \\ &= x^2 + y^2 + (x - y)^2 \geq 0, \end{aligned}$$

which tells that R is an invariant region. The critical points are points that satisfy the equations

$$\begin{aligned} -x + y + xy &= 0 \\ x^2 + y &= 0, \end{aligned}$$

which is $(0, 0)$. Since $(0, 0)$ is not in R, R contains no critical points. Therefore, the system has a periodic solution in R.

Non-Existence of Limit Cycles

As well as existence criterions, there are also non-existence criterions.

Theorem 6.8.3: Bendixson-Dulac Theorem

Suppose R is a simply connected region. If there exists a function $\phi(x, y)$ with continuous first partial derivatives such that

$$\frac{\partial(\phi f)}{\partial x} + \frac{\partial(\phi g)}{\partial y}$$

doesn't change its sign in R, then the autonomous system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

doesn't have a periodic solution in R.

Proof. Assume that there is a periodic solution with period T around a simple closed curve C inside R, and let D be the region bounded by C. Since $\partial(\phi f)/\partial x + \partial(\phi g)/\partial y$ does not change its sign within R, we have

$$\iint_D \left(\frac{\partial(\phi f)}{\partial x} + \frac{\partial(\phi g)}{\partial y} \right) dx\, dy \neq 0.$$

However, by Green's theorem, we get

$$\begin{aligned} \iint_D \left(\frac{\partial(\phi f)}{\partial x} + \frac{\partial(\phi g)}{\partial y} \right) dx\, dy &= \oint_C (\phi f\, dy - \phi g\, dx) \\ &= \int_C \phi(f\, g - g\, f)\, dt = 0, \end{aligned}$$

which is a contradiction. Therefore, the system does not have a periodic solution in R. ■

There aren't specific rules for setting an appropriate function $\phi(x, y)$, and one should construct $\phi(x, y)$ to make $\partial(\phi f)/\partial x + \partial(\phi g)/\partial y$ positive or negative. For some cases, $\phi(x, y) = 1$ would work. The corollary when $\phi(x, y) = 1$ is called the **Bendixson criterion**.

Corollary : Bendixson Criterion

Let f and g be functions such that $\partial f/\partial x$ and $\partial g/\partial y$ are continuous on a region R which is simply connected. If

$$\operatorname{div}\mathbf{V} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

doesn't change its sign in R, then the autonomous system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

doesn't have a periodic solution inside R.

Example 2

Show that the system

$$x' = -xy^2 + x + 2y + 6$$
$$y' = -y^3 + x^2 - 2y$$

does not have a periodic solution.

Solution We have

$$\begin{aligned}\operatorname{div}\mathbf{V} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \\ &= -y^2 + 1 - 3y^2 - 2 \\ &= -4y^2 - 1 < 0.\end{aligned}$$

Since div $\mathbf{V}$ is negative in every point, the system does not have a periodic solution by the Bendixson criterion.

Example 3

Show that the system

$$x' = e^x y - 2x - xy$$
$$y' = y^2 + y$$

does not have a periodic solution.

Solution With $\phi(x,y) = y$, we have

$$\begin{aligned}\frac{\partial(\phi f)}{\partial x} + \frac{\partial(\phi g)}{\partial y} &= e^x y^2 - 2y - y^2 + 3y^2 + 2y \\ &= e^x y^2 + 2y^2 > 0.\end{aligned}$$

Since $\partial(\phi f)/\partial x + \partial(\phi g)/\partial y$ is positive in every point, the system does not have a periodic solution by the Bendixson-Dulac theorem.

Theorem 6.8.4: Critical Point Criterion

If an autonomous system has a periodic solution around a simple closed curve C, then there is a critical point in the interior of C.

Notice that the critical point criterion can also be a cycle criterion.

Corollary : Periodic Solution Criterion

If a simply connected region R does not contain any critical points, then there aren't any periodic solutions in R.

Example 4

Show that the system

$$\begin{aligned}x' &= (x-1)^2 + y^2 \\ y' &= x + y - 2\end{aligned}$$

does not have a periodic solution.

Solution We use the periodic solution criterion. The critical points of the system should satisfy

$$\begin{aligned}(x-1)^2 + y^2 &= 0 \\ x + y - 2 &= 0.\end{aligned}$$

The only point that satisfy the first equation is $(1,0)$. However, this point doesn't satisfy $x+y-2=0$, and there are no critical points to the system. By the periodic solution criterion, there aren't any periodic solutions.

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